ISRAEL JOURNAL OF MATHEMATICS, Vol. 65, No. 3, 1989

MODULI OF NON-DENTABILITY AND THE RADON–NIKODÝM PROPERTY IN BANACH SPACES

BY

W. SCHACHERMAYER,[a] A. SERSOURI[b] AND E. WERNER[c]

[a]*Johannes Kepler University, Linz, Austria*;

[b]*Equipe d'Analyse, Université Paris VI and University of Texas, Austin*; and

[c]*Equipe d'Analyse, Université Paris VI and Oklahoma State University, Stillwater*

ABSTRACT

We show that for a separable Banach space X failing the Radon–Nikodým property (RNP), and $\varepsilon > 0$, there is a symmetric closed convex subset C of the unit ball of X such that every extreme point of the weak-star closure of C in the bidual X^{**} has distance from X bigger than $1 - \varepsilon$. An example is given showing that the full strength of this theorem does not carry over to the non-separable case. However, admitting a renorming, we get an analogous result for this theorem in the non-separable case too. We also show that in a Banach space failing RNP there is, for $\varepsilon > 0$, a convex set C of diameter equal to 1 such that each slice of C has diameter bigger than $1 - \varepsilon$. Some more related results about the geometry of Banach spaces failing RNP are given.

1. Introduction

Let C be a closed convex bounded (abbreviated c.c.b.) subset of a Banach space X and denote by $\tilde{C}$ its weak-star closure in the bidual X^{**}.

It is well known that a Banach space X fails the Radon–Nikodým property (RNP) if and only if there exists a c.c.b. subset of X without denting points (see [D–U]). The failure of RNP for X is also equivalent to the existence of a c.c.b. subset C of X such that all the extreme points of $\tilde{C}$ are in $X^{**} \setminus X$ ([B1], [St2]). By considering the moduli

$$\delta_1(C) = \inf\{\operatorname{diam} S : S \text{ slice of } C\} \quad \text{and} \quad \delta_2(C) = \operatorname{dist}(\operatorname{Ext}(\tilde{C}), X)$$

one can give the following quantitative analogues of the above statements.

Received October 27, 1986 and in final revised form August 25, 1988

THEOREM 1.1. *Let X be a Banach space failing* RNP. *Then*

(i) *for every* $\varepsilon > 0$ *there exists a separable* c.c.b. *subset* C *of* X *such that* $\operatorname{diam} C = 1$ *and* $\delta_1(C) > 1 - \varepsilon$.

(ii) *For every* $\varepsilon > 0$ *there exists a* c.c.b. *separable circled subset* C *of* ball(X) *such that* $\delta_2(C) > (1/2) - \varepsilon$.

(iii) *If moreover* X *is separable, for every* $\varepsilon > 0$ *there exists a* c.c.b. *circled subset* C *of* ball(X) *such that* $\delta_2(C) > 1 - \varepsilon$.

THEOREM 1.2. *Let* C *be a* c.c.b. *subset of* X. *Then*

(i) $\delta_1(C) \geqq \delta_2(C)$.

(ii) *Conversely, if* $\delta_1(C) = \beta > 0$ *then for every* $\varepsilon > 0$, *there is a* c.c.b. *subset* D *of* C *such that* $\delta_2(D) > \beta/4 - \varepsilon$. *Moreover if* X *is separable, then* D *can be chosen so that* $\delta_2(D) > \beta/2 - \varepsilon$.

Note that in the second statement of the above theorem one has to pass to a subset D of C (see Remark 2.8). Moreover the value $\beta/2$ for the separable case is best possible (see Proposition 2.10).

Let us now give an outline of the organization of this paper. Section 2 is devoted to the proofs of Theorems 1.1 and 1.2. In Section 3, we show that for some equivalent norms on X the set C of Theorem 1.1 can be chosen to be the unit ball of these norms.

Analogous results to Theorem 1.1 for w^*-compact subsets of dual Banach spaces X^* failing RNP are discussed in Section 4. It turns out that there holds an analogous result for the existence of "big" slices for these sets while there is no analogue to 1.1(ii). We also give related results for the notions of Point of Continuity property and strong regularity and present a result of M. Talagrand: There is a w^*-compact convex subset K of $\mathcal{M}(\Delta)$ such that every convex combination of *weak* slices of K is big.

In Sections 5, 6 and 7 we discuss several counter-examples. In Section 5 we prove that for the predual J_*T of the James' tree space there exists a constant $\beta < 2$ for which every c.c.b. subset of the unit ball of J_*T satisfies $\delta_1(C) \leqq \beta$. This shows that the convex C constructed in Theorem 1.1(i) cannot be in general a subset of a ball of radius 1/2.

Section 6 is devoted to proving that for some constant $\alpha < 1$, every c.c.b. subset C of the unit ball of JT^* satisfies $\delta_2(C) \leqq \alpha$. This shows that the full strength of Theorem 1.1(iii) cannot be carried over to the non-separable case.

Finally in Section 7, we construct a subspace Z_A of JT^* which shows that the separability assumption in the measurability Lemma 7.1, used in the analytical proof of Theorem 1.1(iii), is essential. The space Z_A also answers negatively a question of K. Musiał about extendability of Pettis integrable functions.

We recall now some definitions: A slice of C is a subset S of C of the form $S(x^*, \delta) = \{x \in C : x^*(x) > M_{x^*} - \delta\}$, where $x^* \in X^*$, $\| x^* \| = 1$, $\delta > 0$ and $M_{x^*} = \sup_{x \in C} x^*(x)$.

A Banach space X has RNP if every operator $T : L^1[0, 1] \to X$ is representable, i.e., there exists a Bochner integrable function $f : [0, 1] \to X$ such that $Tg = \int g f$ for every $g \in L^1[0, 1]$.

For $A \subseteq [0, 1]$, $m(A) > 0$, m the Lebesgue measure on $[0, 1]$, let $\mathcal{F}_A = \{f \in L^1[0, 1] : f = f \cdot \chi_A,\ f \geqq 0,\ \| f \|_{L_1} = 1\}$. $\mathcal{F}_{[0,1]}$ will simply be denoted by $\mathcal{F}$. (Let us point out that the systematic use of the sets $\mathcal{F}_A$ was initiated by C. Stegall [St2].)

The two last-named authors want to thank Professors B. Maurey and G. Godefroy for helpful discussions. Special thanks go to M. Talagrand for allowing us to include an unpublished result of his (Theorem 4.6 below).

2. Proofs of Theorems 1.1 and 1.2

To prove Theorem 1.1 we need the following lemmas.

LEMMA 2.1. *Let $T : L^1[0, 1] \to X$ be a bounded linear operator, $A \subseteq [0, 1]$, $m(A) > 0$, and $C = \overline{T(\mathcal{F}_A)}$. If S is a slice of C then there is $B \subseteq A$, $m(B) > 0$ such that $S \supseteq \overline{T(\mathcal{F}_B)}$.*

PROOF. S is given by some $x^* \in X^*$, $\| x^* \| = 1$ and $\delta > 0$ via

$$S = S(x^*, \delta) = \{x \in C : \langle x, x^* \rangle > M_{x^*} - \delta\}$$

where $M_{x^*} = \sup_{y \in C} \langle y, x^* \rangle$.

One easily verifies that $M_{x^*} = \operatorname{ess\,sup} \chi_A \cdot (T^* x^*)$.

The set $B = \{t \in A : T^*(x^*)(t) > M_{x^*} - \delta/2\}$ satisfies the requirement of the lemma. ■

LEMMA 2.2. *Let $T : L^1[0, 1] \to X$ be a bounded, linear operator, $A \subseteq [0, 1]$, $m(A) > 0$, $\alpha > 0$ and $x \in X$.*

If $T(\mathcal{F}_A)$ is not contained in the closed ball of radius α around x then there is $B \subseteq A$, $m(B) > 0$ such that $\operatorname{dist}(x, T(\mathcal{F}_B)) > \alpha$.

PROOF. If $T(\mathcal{F}_A)$ is not contained in $\operatorname{ball}(x, \alpha)$ there is a slice S of $T(\mathcal{F}_A)$ such that $\operatorname{dist}(x, S) > \alpha$. Lemma 2.1 then furnishes the desired B. ■

The next lemma is a well-known folklore result. We include a proof for the sake of completeness.

LEMMA 2.3. *Let X be a Banach space, Y be a subspace of X and $\xi \in Y^{**} = \bar{Y}^{\sigma(X^{**},X^{*})}$. Then*

$$\tfrac{1}{2}\operatorname{dist}(\xi, Y) \leqq \operatorname{dist}(\xi, X) \leqq \operatorname{dist}(\xi, Y).$$

PROOF. Let $x \in X$. By the Hahn–Banach theorem it is easily seen that $\operatorname{dist}(x, Y) = \operatorname{dist}(x, Y^{**})$. Let $\xi \in Y^{**}$. We put $a = \operatorname{dist}(\xi, X)$. Let $\eta > 0$ be given. Then there exists $x_\eta \in X$ such that $\| x_\eta - \xi \| < a + \eta$. By the above remark there exists $y_\eta \in Y$ such that $\| x_\eta - y_\eta \| < a + \eta$. Therefore $\| \xi - y_\eta \| < 2a + 2\eta$ and the lemma is proved. ∎

PROOF OF THEOREM 1.1. (i) If X fails RNP then there is a non-representable operator $T: L^1[0, 1] \to X$. Hence there is $A \subseteq [0, 1]$, $m(A) > 0$ and $\alpha > 0$ such that $\operatorname{diam}(T(\mathscr{F}_B)) \geqq \alpha$ for every $B \subseteq A$, $m(B) > 0$. Indeed, otherwise a standard exhaustion argument would show that T is representable.

Let $\beta = \inf\{\operatorname{diam} T(\mathscr{F}_B) : B \subseteq A, m(B) > 0\}$, and $B_0 \subseteq A$, $m(B_0) > 0$, such that $\beta_0 = \operatorname{diam} T(\mathscr{F}_{B_0}) < \beta/(1 - \varepsilon)$.

The set $C = \overline{T(\mathscr{F}_{B_0})}/\beta_0$ satisfies the requirements of 1.1(i) in view of Lemma 2.1.

(ii) The assertion of 1.1(ii) follows immediately from 1.1(iii) and Lemma 2.3 (take a separable subspace failing RNP).

(iii) As in the proof of Theorem 1.1(i) let $T: L^1[0, 1] \to X$ be a non-representable operator and apply exhaustion (compare [St2]) to obtain $A \subseteq [0, 1]$, $m(A) > 0$, and $\alpha > 0$ such that for every $y \in X$ and $B \subseteq A$, $m(B) > 0$,

$$T(\mathscr{F}_B) \not\subseteq \operatorname{ball}(y, \alpha) = \{z \in X : \| z - y \| \leqq \alpha\}.$$

Let $\beta = \inf_{y \in X} \inf_{B \subseteq A, m(B) > 0} \sup_{D \subseteq B, m(D) > 0} \operatorname{dist}(y, T(\mathscr{F}_D))$ and find y_0 and B_0 such that

$$\sup_{D \subset B_0,\, m(D) > 0} \operatorname{dist}(y_0, T(\mathscr{F}_D)) = \beta_0 < \beta/(1 - \varepsilon/2).$$

It follows from Lemma 2.2 that $T(\mathscr{F}_{B_0})$ is contained in $\operatorname{ball}(y_0, \beta_0)$.

Let $(y_n)_{n=1}^\infty$ be a dense sequence in X. For $n \in \mathbf{N}$ find — again by exhaustion and using 2.2 — subsets $D_1^n, \ldots, D_{N(n)}^n$ of B_0 such that

$$\operatorname{dist}(y_n, T(\mathscr{F}_{D_i^n})) > \beta(1 - \varepsilon/2) \qquad \text{for } 1 \leqq i \leqq N(n)$$

and

$$m\left(\bigcup_{i=1}^{N(n)} D_i^n\right) > m(B_0)(1-2^{-(n+2)}).$$

Let $E = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N(n)} D_i^n$ and let $E_i^n = D_i^n \cap E$. We shall show that, for $n \in \mathbf{N}$, every extreme point y^{**} of the weak-star closure $\widetilde{T(\mathscr{F}_E)}$ of $T(\mathscr{F}_E)$ has distance from y_n greater than $\beta(1-\varepsilon/2)$.

Indeed, for every $n \in \mathbf{N}$, $\mathscr{F}_E = \text{convex hull}\{\mathscr{F}_{E_1^n}, \ldots, \mathscr{F}_{E_{N(n)}^n}\}$ and — passing to the weak-star closures in $L^1[0,1]^{**}$ — we get

$$\tilde{\mathscr{F}}_E = \text{convex hull}\{\tilde{\mathscr{F}}_{E_1^n}, \ldots, \tilde{\mathscr{F}}_{E_{N(n)}^n}\}.$$

As $\widetilde{T(\mathscr{F}_E)} = T^{**}(\tilde{\mathscr{F}}_E) = \text{convex hull}\{\widetilde{T(\mathscr{F}_{E_i^n})}: 1 \leqq i \leqq N(n)\}$, we see that the extreme point y^{**} of $\widetilde{T(\mathscr{F}_E)}$ must be contained in some $\widetilde{T(\mathscr{F}_{E_i^n})}$, hence

$$\text{dist}(y_n, y^{**}) \geqq \text{dist}(y_n, \widetilde{T(\mathscr{F}_{E_i^n})}) = \text{dist}(y_n, T(\mathscr{F}_{E_i^n})) > \beta(1-\varepsilon/2).$$

Letting $C_1 = ((1-\varepsilon/2)/\beta)\cdot\overline{[T(\mathscr{F}_E) - y_0]}$ we obtain a closed convex subset of the unit ball of X such that $\delta_2(C_1) \geqq (1-\varepsilon/2)^2 \geqq 1-\varepsilon$.

Letting $C = \text{closed convex hull}(C_1, -C_1)$ one obtains $\tilde{C} = \text{convex hull}(\tilde{C}_1, -\tilde{C}_1)$. Hence every extreme point of $\tilde{C}$ has distance from X greater than $1-\varepsilon$, as it must be an extreme point of either $\tilde{C}_1$ or $-\tilde{C}_1$. ■

To prove Theorem 1.2 we need one more lemma which is a quantitative version of a result of R. Huff and P. Morris ([D–U], VII. 4.1).

LEMMA 2.4. *Let C be a* c.c.b. *subset of a Banach space X, then*

(i) *If C is such that every slice has diameter bigger than* 1, *then there is no slice of C which can be covered by finitely many sets of diameter strictly less than* 1.

(ii) *If C is such that for every* ball B *of radius less than* 1 *one has $C = \overline{\text{conv}}(C \setminus B)$, then for every finite number $B_1, \ldots, B_n$ of balls of radius less than* 1 *one has $C = \overline{\text{conv}}(C \setminus \bigcup_{i=1}^n B_i)$.*

PROOF. The proof will use some arguments and concepts due to J. Bourgain [B2]. If $S = S(x^*, \delta)$ is a slice of C we denote by $E^0(\tilde{S}) = \{\xi \in \text{Ext}(\tilde{C}) : \xi(x^*) > M_{x^*} - \delta\}$.

(i) Suppose that there is a slice S of C, and finitely many sets $A_1, \ldots, A_n$ of X all of diameter strictly less than 1 such that $S \subset \bigcup_{i=1}^n A_i$. This inclusion obviously implies $E^0(\tilde{S}) \subset \bigcup_{i=1}^n \tilde{A}_i$.

Without loss of generality we may assume that $(A_i)_{i \leq n}$ is minimal in the sense that no strict subfamily of $(\tilde{A}_i)_{i \leqq n}$ covers $E^0(\tilde{S})$. Hence there exists $\xi \in E^0(\tilde{S})$ such that $\xi \in \tilde{A}_1$ and $\xi \notin \tilde{A}_i$, for $i \geqq 2$.

Since ξ is an extreme point of $\tilde{C}$, we can find a w^*-slice T of C such that $\xi \in \tilde{T} \subset \tilde{S}$ and $\tilde{T} \cap \tilde{A}_i = \varnothing$ for every $i \geqq 2$. (We use the fact that the w^*-slices containing ξ form a basis for the w^*-topology of $\tilde{C}$ at ξ.) Hence

$$E^0(\tilde{T}) \subset \tilde{T} \cap E^0(\tilde{S}) \subset \tilde{A}_1.$$

By a result of J. Bourgain ([B3], Lemma 3) there is a slice R of C such that $\tilde{R} \subset \tilde{T}$, and for every $x \in R$ one has

$$\operatorname{dist}(x, \overline{\operatorname{conv}}^*(E^0(\tilde{T}))) < \frac{1 - \operatorname{diam}(A_1)}{4}.$$

This implies

$$\operatorname{diam}(R) < \operatorname{diam}(E^0(\tilde{T})) + \frac{1 - \operatorname{diam}(A_1)}{2} \leqq \frac{1 + \operatorname{diam}(A_1)}{2} < 1.$$

This finishes the proof of (i).

(ii) The proof of part (ii) is similar to the above one if we observe that the conclusion of (ii) is equivalent to the fact that no slice of C can be covered by finitely many balls of radius less than 1. ■

Proof of Theorem 1.2. (i) Let C be a c.c.b. subset of a Banach space X. If S is a slice of C, then $\tilde{S}$ contains an extreme point ξ of $\tilde{C}$. As $\tilde{S} \cap X \neq \varnothing$ we get

$$\operatorname{diam} S = \operatorname{diam} \tilde{S} \geqq \operatorname{dist}(\xi, X) \geqq \delta_2(C),$$

hence $\delta_1(C) \geqq \delta_2(C)$.

(ii) Let us first suppose that X is separable. Fix $\varepsilon > 0$, and let $(y_n)_{n \geqq 1}$ be a dense sequence in X. Using the above lemma, and a standard perturbation argument ([D–U], proof of V.3.4) one can show that there exists a finite C-valued martingale $(M_n)_{n \geqq 1}$ on $[0, 1]$ such that

$$\| M_{n+1}(t) - y_j \| > (\beta - \varepsilon)/2 \qquad \text{for every } t \in [0, 1], \quad n \in \mathbf{N}, \quad 1 \leqq j \leqq n.$$

If $T : L^1[0, 1] \to X$ is the operator associated to the martingale $(M_n)_{n \geqq 1}$, it is not difficult to see that for $A \subseteq [0, 1]$, $m(A) > 0$ and $n \geqq 1$, $T(\mathcal{F}_A)$ is not contained in ball$(y_n, (\beta - \varepsilon)/2)$. The proof is continued as in Theorem 1.1(iii) by considering the expression

$$\inf_n \inf_{B \subset A, m(B) > 0} \sup_{D \subset B, m(D) > 0} \operatorname{dist}(y_n, T(\mathcal{F}_D)) \geqq (\beta - \varepsilon)/2.$$

The non-separable case follows from the separable one and Lemma 2.3. ■

REMARK 2.5. In view of Theorems 1.1(i) and 1.2(ii) one might ask whether, given a separable Banach space X and $\varepsilon > 0$, it is possible to find a convex closed subset $C \subset B(X)$ of the unit ball of X such that all the slices of S have diameter at least $2 - \varepsilon$. We will see in Section 5 that such a result is not true in J_*T, the predual of the James tree space. However the answer is positive for Banach lattices [W].

REMARK 2.6. It is not possible, in general, to take $\varepsilon = 0$ in Theorem 1.1(iii) as the following equivalent norm on c_0 shows:

$$|(x_n)| = \sup_n |x_n| + \sum_{n=1}^{\infty} \frac{|x_n|}{2^n} .$$

Its bidual norm on l^∞ is given by the same expression and for every $\xi \in l^\infty \setminus \{0\}$ one has $\mathrm{dist}_{|\cdot|}(\xi; c_0) < |\xi|$. Indeed, let n be such that $\xi_n \neq 0$. Then $|\xi - \xi_n e_n| < |\xi|$.

REMARK 2.7. The question whether one may drop the separability assumption in Theorem 1.1(iii) is quite delicate and will be settled negatively in 6.1 below.

REMARK 2.8. The following example shows that in Theorem 1.2(ii) we have to pass to a subset D of C and that it is not possible — in general — to get a value better than $\beta/2$ in the conclusion of 1.2(ii).

All the slices of the unit ball of $C[0, 1]$ have diameter equal to 2, while the constant function $\chi_{[0,1]}$ is an extreme point of the unit ball of $C[0, 1]^{**}$. (See also Remark 2.9.)

Let us sketch a proof of these two claims:

(1) Let $\mu \in \mathscr{M}[0, 1]$, $\|\mu\| = 1$. For $\delta > 0$ choose two non-void open sets A and B such that $A \cap B = \varnothing$ and $|\mu|(B) > 1 - \delta/3$, and a function $f_0 \in C[0, 1]$, $\|f_0\| = 1$ such that $\mu(f_0) > 1 - \delta/3$.

Choose now two functions $f_1, f_2 \in C[0, 1]$, $\|f_i\| = 1$ such that $f_1 = 1$ (resp. $f_2 = -1$) on A and $f_1 = f_2 = f_0$ on B. One quickly verifies that $\mu(f_1) > 1 - \delta$ and $\mu(f_2) > 1 - \delta$ and it is clear that $\|f_1 - f_2\| = 2$. Hence the diameter of the slice $S(\mu, \delta)$ of the unit ball of $C[0, 1]$ is equal to 2.

(2) Let L be the compact set such that $C[0, 1]^{**} = C(L)$ and denote by $\psi: L \to [0, 1]$ the quotient map obtained by restricting the elements of $C(L)^*$

to $C[0,1]$. Then the natural injection $i_0: C[0,1] \to C(L)$ is given by $i_0(f) = f \circ \psi$, hence $i_0(\chi_{[0,1]}) = \chi_L$ which is an extreme point of the unit ball of $C(L)$.

REMARK 2.9. One might also ask, whether — in the situation of Theorem 1.1(iii) — one can obtain a closed convex set C of diameter less than 1 such that $\delta_2(C) \geqq (1-\varepsilon)$. This is not true as shown by the next proposition. Hence, "a subset of the unit ball" is the proper assumption for Theorem 1.1(iii) and a "set of diameter 1" the proper one for Theorem 1.1(i).

PROPOSITION 2.10. *Every bounded subset C of $L^1[0,1]$ satisfies $\delta_1(C) \geqq 2\delta_2(C)$.*

PROOF. Indeed, following [G–M] we define

$$\rho(C) = \lim_{M\to\infty} \sup_{f\in C} \int_{|f|\geqq M} |f|\,dt.$$

Then we see that $\sup_{x^{**}\in\tilde{C}} \operatorname{dist}(x^{**}, L^1) \leqq \rho(C)$, while $\operatorname{diam}(C) \geqq 2\rho(C)$. [These facts about the "modulus of equi-integrability" are well-known.]

The conclusion of the proposition follows by applying these inequalities to the slices S of C, and by noting that for every slice S of C, $\tilde{S} \cap \operatorname{Ext}(\tilde{C}) \neq \varnothing$. ■

REMARK 2.11. The value 1/2 in Lemma 2.3 is optimal. The subsequent easy example will be crucial as motivation for 6.1 and 7.2 below.

Let $\bar{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ be the Alexandroff compactification of $\mathbf{N}$, c the Banach space of continuous functions on $\bar{\mathbf{N}}$ and c_0 the hyperplane of c defined by

$$c_0 = \{f \in c : f(\infty) = 0\}.$$

Note that c^{**} equals $l^\infty(\bar{\mathbf{N}})$ while c_0^{**} equals the hyperplane of $l^\infty(\bar{\mathbf{N}})$ given by

$$c_0^{**} = \{f \in l^\infty(\bar{\mathbf{N}}) : f(\infty) = 0\}.$$

Let $f_0 \in c_0{}^{**}$ be given by $f_0(n) = 1$ for every $n \in \mathbf{N}$ and $f_0(\infty) = 0$. Clearly $\operatorname{dist}(f_0, c_0) = 1$. However, $\operatorname{dist}(f_0, c) = 1/2$. Indeed, the function g_0 given by $g_0(n) = 1/2$ for every $n \in \bar{\mathbf{N}}$ is in c and $\| f_0 - g_0 \| = 1/2$, whence $\operatorname{dist}(f_0, c) \leqq 1/2$. The reverse inequality $\operatorname{dist}(f_0, c) \geqq 1/2$ is given by 2.3.

3. Two renorming results

In this section we will prove that for spaces failing RNP, one can find equivalent norms for which the new unit balls are "bad" with respect to the phenomena of "big slices" and of "far extreme points."

THEOREM 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and $\alpha > 0$. Suppose there exists an absolutely convex, closed subset C of $B_{\|\cdot\|}(X)$ such that all slices of C have $\|\cdot\|$-diameter $\geqq \alpha$. Then for every $\varepsilon > 0$ there exists an equivalent norm $|\cdot|$ on X such that $|\cdot| \geqq \|\cdot\|$ and every slice S of $B_{|\cdot|}(X)$ satisfies*: $\mathrm{diam}_{|\cdot|}(S) \geqq \mathrm{diam}_{\|\cdot\|}(S) \geqq \alpha - \varepsilon$.

PROOF. Fix $\varepsilon > 0$ and let C be as in the assumption.

We define an equivalent norm $|\cdot|$ on X by

$$B_{|\cdot|}(X) = \frac{1}{1+\varepsilon}\overline{(C + \varepsilon B_{\|\cdot\|}(X))}.$$

We will prove that every slice of $B_{|\cdot|}(X)$ has $\|\cdot\|$-diameter at least $(\alpha - 2\varepsilon)/(1+\varepsilon)$, so every slice of $B_{|\cdot|}(X)$ has also $|\cdot|$-diameter at least $(\alpha - 2\varepsilon)/(1+\varepsilon)$.

Indeed, let $f \in X^*$, $|f| = 1$, $\beta > 0$. We put $\gamma = \sup_C f$ and $\delta = \sup_{B_{|\cdot|}(X)} f$. As easy computation shows that $(\gamma + \varepsilon\delta)/(1+\varepsilon) = 1$ and

$$\frac{1}{1+\varepsilon}[S(C; f, (1+\varepsilon)\beta/2) + \varepsilon \cdot S(B_{\|\cdot\|}(X); f, (1+\varepsilon)\beta/2\varepsilon)] \subseteq S(B_{|\cdot|}(X); f, \beta).$$

From this we deduce that $\|\cdot\|$-$\mathrm{diam}[S(B_{|\cdot|}(X); f, \beta)] \geqq (\alpha - 2\varepsilon)/(1+\varepsilon)$. ∎

COROLLARY 3.2. *Let X be a Banach space without* RNP. *Then for every $\varepsilon > 0$ there exists an equivalent norm $|\cdot|$ on X such that every slice S of $B_{|\cdot|}(X)$ satisfies* $\mathrm{diam}_{|\cdot|}(S) \geqq 1 - \varepsilon$.

PROOF. In view of the preceeding theorem it is enough to find a c.c.b. symmetric subset C of Ball(X) with $\delta_1(C) \geqq 1 - \varepsilon$. Such a set exists by Theorems 1.1(iii) and 1.2(i) applied to a separable subspace Y of X failing RNP. ∎

We will see in Section 6 that Theorem 1.1(iii) is false in the non-separable case, however allowing a renorming there is an analogue for it.

THEOREM 3.3. *Let X be a Banach space without* RNP. *Then for every $\varepsilon > 0$ there exists an equivalent norm $|\cdot|$ on X such that* $\mathrm{dist}_{|\cdot|}(\mathrm{Ext}(B_{|\cdot|}(X^{**})); X) \geqq 1 - \varepsilon$.

PROOF. Let X be a Banach space without RNP, Y a separable subspace of X without RNP.

For $0 < \varepsilon < 1/2$, let $C \subseteq B(Y)$ be the absolutely convex closed subset given by Theorem 1.1(iii). We define an equivalent norm $[\![\cdot]\!]$ on X by

$$B_{[\![\cdot]\!]}(X) = \overline{B(Y) + \frac{\varepsilon}{2(1-2\varepsilon)} B(X)}.$$

Then $C \subseteq B_{[\![\cdot]\!]}(X)$ and $\text{dist}_{[\![\cdot]\!]}(\text{Ext}(\hat{C}), X) \geqq 1 - 2\varepsilon$.

Indeed, let $\xi \in \text{Ext}(\tilde{C})$. We put $L = \xi + (1-2\varepsilon)B(Y^{**})$.

From $\text{dist}_{\|\cdot\|}(\xi, Y) \geqq 1 - \varepsilon$, we deduce that $\text{dist}_{\|\cdot\|}(L, Y) \geqq \varepsilon$. By Lemma 2.3 it follows that $\text{dist}_{\|\cdot\|}(L, X) \geqq \varepsilon/2$. This means that

$$\xi - x \notin \left[(1-2\varepsilon)B(Y^{**}) + \frac{\varepsilon}{2} B(X^{**})\right] = (1-2\varepsilon)B_{[\![\cdot]\!]}(X^{**}) \qquad \text{for every } x \in X.$$

This means that $[\![\xi - x]\!] \geqq 1 - 2\varepsilon$ for every $x \in X$.

The set $B_{|\cdot|}(X) = \overline{C + \varepsilon B_{[\![\cdot]\!]}(X)}$ is the unit ball of an equivalent norm $|\cdot|$ on X which satisfies $\text{dist}_{|\cdot|}(\text{Ext}(B_{|\cdot|}(X^{**})); X) \geqq 1 - 4\varepsilon$. ■

REMARK 3.4. Observe that none of the conclusions, neither of Corollary 3.2 nor of Theorem 3.3, is true for every norm on X as is shown by the following observation.

If $\|\cdot\|$ is any norm on X, let $x_0 \in X$ such that $\|x_0\| = 2$, and define a new norm $|\cdot|$ by $B_{|\cdot|}(X) = \overline{cv}[\pm x_0, B_{\|\cdot\|}(X)]$, then x_0 is a strongly exposed point of $B_{|\cdot|}(X)$ as one quickly verifies.

PROBLEM 3.5. Given a Banach space X failing RNP, and $\varepsilon > 0$, does there exist an equivalent norm $|\cdot|$ on X such that all the slices of the new unit ball have $|\cdot|$-diameter at least $2 - \varepsilon$?

Notice that in view of Theorem 1.2, a positive answer to this problem will give an improvement of the above two results. Let us also point out that the answer is positive for non-strongly regular dual spaces ([Go], [Go–M]) and for Banach lattices failing RNP [W].

4. Related results

In this section we prove that for a dual Banach space X^* failing RNP, an analogous result for Theorem 1.1(i) holds true for w^*-compact sets. On the other hand the w^*-version of Theorem 1.1(ii) fails to be true, i.e., the set C cannot be taken to be w^*-compact in general. This solves negatively a problem of J. Diestel and J. J. Uhl. (See Remark 4.5.)

We end this section by giving analogues of Theorem 1.1(i) for the notions of point of continuity property (PCP) and strong regularity.

PROPOSITION 4.1. *Let X^* be a dual Banach space failing* RNP. *For* $\varepsilon > 0$, *there exists a convex w^*-compact set C of X^* such that* $\operatorname{diam}(C) = 1$ *and all the w^*-slices of C have diameter greater than* $1 - \varepsilon$.

PROOF. By a result of C. Stegall ([St1], Theorem 1) there is $T_0 : X \to L^\infty(\Delta, \mu)$ such that the restriction of T_0^* to $L^1(\Delta)$ fails to be representable. Here Δ denotes the Cantor group $\{-1, +1\}^{\mathbf{N}}$ and μ the Haar measure on Δ.

As in Lemma 2.1, we can prove that if $A \subset \Delta$, $\mu(A) > 0$, then for every w^*-slice S of $\overline{T(\mathscr{F}_A)}^*$ there exists $B \subset A$, $\mu(B) > 0$ such that $\overline{T(\mathscr{F}_B)}^* \subset S$. Hence the rest of the proof of 1.1(i) carries over verbatim. ■

PROPOSITION 4.2. *There exists a separable Banach space X such that X^* does not have* RNP, *and such that every convex w^*-compact subset C of X^* contains an extreme point of $\tilde{C}$ (its w^*-closure in X^{***}).*

To prove this proposition we use the notion of furthest point.

Let K be a bounded convex subset of a Banach space X and $x \in X$. We say that x is a furthest point w.r. to K if there exists $y_0 \in K$ such that $\| x - y_0 \| = \sup_{y \in K} \| x - y \|$.

Let us prove the following lemma.

LEMMA 4.3. *If X is a Banach space with* RNP *and such that the weak and the norm topologies coincide on the unit sphere of X^*, then every convex w^*-compact subset K of X^* has a denting point. In particular* $K \cap \operatorname{Ext}(\tilde{K}) \neq \emptyset$.

PROOF. Let K be a convex w^*-comapct subset of X^*. Since X has RNP, by a result of Deville and Zizler ([D–Z], Prop. 3) K has a furthest point say x_0^*. This means that the convex w^*-lower semi-continuous function $\varphi(y^*) = \| y^* - x_0^* \|$ achieves its maximum on K, and since K is convex and w^*-compact, φ achieves its maximum on some extreme point y_0^* of K (immediate consequence of Choquet's representation theorem).

Now since the weak and norm topologies coincide on the unit sphere $S(X^*)$ of X^*, all the points of $S(X^*)$ are points of continuity of $\mathrm{Id} : (B(X^*), w) \to (B(X^*), \| \cdot \|)$. And since $K \subset \operatorname{ball}(x_0^*, \| y_0^* - x_0^* \|)$, y_0^* is a point of continuity of $\mathrm{Id} : (K, w) \to (K, \| \cdot \|)$, and since y_0^* is an extreme point of K, it is also a denting point of K by a result of [L–L–T]. ■

PROOF OF PROPOSITION 4.2. It is shown in [S1] that the weak and norm topologies coincide on the unit sphere $S(JT^*)$ of JT^*, the dual of the James' tree space.

Since JT has RNP, and JT^* does not have RNP, Proposition 4.2 follows from Lemma 4.3. ∎

REMARK 4.4. Let us also mention that in [S1] the following more precise version of 4.2 is shown: If C is convex, w^*-compact in JT^* then the functionals which strongly expose C form a dense G_δ subset of JT^{**}.

REMARK 4.5. Diestel and Uhl asked whether X^* has RNP or X does not contain l^1 if we assume that every convex w^*-compact subset of X^* has weak slices of arbitrarily small diameter. Proposition 4.2 gives a negative answer to the first part of the question. The second part of the question was recently solved positively for separable Banach spaces by M. Talagrand. We include this result and thank M. Talagrand for permitting us to do so.

THEOREM 4.6 [Talagrand]. *There exists a convex w^*-compact subset K of $\mathbf{P}_\Delta$, the set of probability measures on Δ, such that for every $k \in \mathbf{N}$ and every weak slices $S_1, \ldots, S_k$ of K, one has*

$$\operatorname{diam}\left[\frac{1}{k}(S_1 + \cdots + S_k)\right] = 2.$$

PROOF. We first introduce some notations. For every natural number $s \geqq 3$, we associate the following family of probability measures. (The underlying measure space will be always clear from the context.)

For $I \subset \mathbf{N}$, $i \in \mathbf{N}$, define on the ith copy of $\{0, 1\}$ a measure

$$\nu_{s,I}(i) = \begin{cases} \dfrac{1}{s}\,\delta_0^{(i)} + \dfrac{s-1}{s}\,\delta_1^{(i)} & \text{if } i \in I, \\[2ex] \dfrac{s-1}{s}\,\delta_0^{(i)} + \dfrac{1}{s}\,\delta_1^{(i)} & \text{if } i \notin I. \end{cases}$$

Now for $J \subset \mathbf{N}$, $I \subset J$, and $p \in J$, define

$$\mu_{s,I}^{J} = \bigotimes_{i \in J} \nu_{s,I}(i),$$

$$\bar{\mu}_{s,I}^{J,p} = \delta_0^{(p)} \otimes \Bigg[\bigotimes_{\substack{i \in J \\ i \neq p}} \nu_{s,I}(i) \Bigg],$$

$$\bar{\bar{\mu}}_{s,I}^{J,p} = \delta_1^{(p)} \otimes \Bigg[\bigotimes_{\substack{i \in J \\ i \neq p}} \nu_{s,I}(i) \Bigg].$$

LEMMA 4.7. *For every $s \geqq 3$, there exists $A(s) \in \mathbf{R}^+$ such that*

$$\alpha_{s,n} = \left\| \sum_{p=1}^{n} \bar{\mu}_{s,[1,n]}^{[1,n],p} - \bar{\bar{\mu}}_{s,[1,n]}^{[1,n],p} \right\|_{\mathcal{M}(\Delta_{[1,n]})} \leqq A(s)\sqrt{n}.$$

REMARK. It might be helpful for the understanding of the lemma to observe that the case $s = 2$ corresponds to the easy implication of the Khintchin-inequality for Rademacher-functions.

PROOF. For $\varepsilon \in \Delta_n$, let $\sigma_0(\varepsilon) = \mathrm{Card}\{j : \varepsilon_j = 0\}$, $\sigma_1(\varepsilon) = \mathrm{Card}\{j : \varepsilon_j = 1\}$. [Note: $\sigma_0(\varepsilon) + \sigma_1(\varepsilon) = |\varepsilon|$.] Then

$$\begin{aligned}
\alpha_{s,n} &= \sum_{\varepsilon \in \Delta_n} \left| \sigma_0(\varepsilon) \cdot \left(\frac{1}{s}\right)^{\sigma_0(\varepsilon)-1} \cdot \left(\frac{s-1}{s}\right)^{\sigma_1(\varepsilon)} - \sigma_1(\varepsilon) \cdot \left(\frac{1}{s}\right)^{\sigma_0(\varepsilon)} \cdot \left(\frac{s-1}{s}\right)^{\sigma_1(\varepsilon)-1} \right| \\
&= \left(\frac{s-1}{s}\right)^{n-1} \sum_{k=0}^{n} C_n^k (s-1)^{-k} |n - sk| \\
&= \left(\frac{s-1}{s}\right)^{n-1} \beta_n .
\end{aligned}$$

By Gauss' formula, a direct computation shows that for every $n \in \mathbf{N}$, and every $0 \leqq l \leqq s - 1$, we have

$$\beta_{sn+l+1} = \frac{s}{s-1} \beta_{sn+l} + 2\left(1 - \frac{l}{s-1}\right)(s-1)^{-n} C_{ns+l}^{n} .$$

Now by Stirling's formula we deduce that

$$\beta_{n+1} \leqq \frac{s}{s-1} \beta_n + \frac{C}{\sqrt{n}} \left(\frac{s}{s-1}\right)^n$$

for some constant C (depending on s), hence $\alpha_{s,n+1} \leqq \alpha_{s,n} + (C/\sqrt{n})$, and then

$\alpha_{s,n} \leqq A\sqrt{n}$. ■

Observe that for every $J \subset \mathbf{N}$, every $I \in \{\phi, J\}$, every finite subset K of J, we have

$$\left\| \sum_{p \in K} \bar{\mu}_{s,I}^{J,p} - \bar{\bar{\mu}}_{s,I}^{J,p} \right\|_{\mathcal{M}(\Delta_J)} = \alpha_{s,|K|}.$$

Lemma 4.8. *For every $s \geqq 3$, every $\alpha > 0$, there exists $n = n(s, \alpha)$ such that: For every $k \in \mathbf{N}$, every $J \subset \mathbf{N}$, $|J| \geqq kn$, every $(\varphi_i)_{1 \leqq i \leqq k} \in \mathcal{M}(\Delta_J)^*$, $\|\varphi_i\| \leqq 1$, there exists $p \in J$ such that*

$$\sup_{I \in \{\phi, J\}} \sup_{1 \leqq i \leqq k} |\varphi_i(\bar{\mu}_{s,I}^{J,p} - \bar{\bar{\mu}}_{s,I}^{J,p})| < \alpha.$$

Proof. Let n be such that $A(s) < \alpha\sqrt{n/4}$, where $A(s)$ is the constant appearing in Lemma 4.7. We shall show that n satisfies the conclusion of Lemma 4.8.

If not, there exists $k \in \mathbf{N}$, $J \subset \mathbf{N}$, $|J| \geqq kn$, $(\varphi_i)_{1 \leqq i \leqq k} \in \mathcal{M}(\Delta_J)^*$, $\|\varphi_i\| \leqq 1$, such that for all $p \in J$

$$\sup_{I \in \{\phi, J\}} \sup_{1 \leqq i \leqq k} |\varphi_i(\bar{\mu}_{s,I}^{J,p} - \bar{\bar{\mu}}_{s,I}^{J,p})| \geqq \alpha.$$

By a cardinality argument, there exist $i_0 \in [1, k]$, $I_0 \in \{\phi, J\}$, $K \subset J$, $|K| \geqq n/4$, and $\varepsilon = \pm 1$ such that

$$\varepsilon\varphi_{i_0}(\bar{\mu}_{s,I_0}^{J,p} - \bar{\bar{\mu}}_{s,I_0}^{J,p}) \geqq \alpha \qquad \text{for every } p \in K.$$

This implies

$$\alpha|K| \leqq \left|\varphi_{i_0}\left(\sum_{p \in K} \bar{\mu}_{s,I_0}^{J,p} - \bar{\bar{\mu}}_{s,I_0}^{J,p}\right)\right| \leqq A(s)\sqrt{|K|}$$

which contradicts the choice of n. ■

Proof of Theorem 4.6. Let $(\mathbf{N}_s)_{s \geqq 3}$ be a partition of $\mathbf{N}$ into *infinite* sets, i.e., $|\mathbf{N}_s| = \infty$, $\forall s \geqq 3$. We define for every $I \subset J \subset \mathbf{N}$ probability measures ρ_I^J on Δ_J by

$$\rho_I^J = \bigotimes_{s \geqq 3} \mu_{s, I \cap \mathbf{N}_s}^{J \cap \mathbf{N}_s}.$$

We also consider the operator $T_J : C(\Delta_J) \to C(\Delta_J)$ defined by $T_J(f)(I) = \rho_I^J(f)$, for every $I \subset J$, where we have identified Δ_J with the power set $\mathcal{P}(J)$.

One can easily check that $T_J^* : \mathcal{M}(\Delta_J) \to \mathcal{M}(\Delta_J)$ satisfies $T_J^*(\delta_I^J) = \rho_I^J$, where

δ_I^J is the Dirac measure at I on Δ_J, and then $T_J^*(\theta) = w^* - \int_{\Delta_J} \rho_I^J \, d\theta(I)$. In particular $T_J^*(\theta)$ is a probability measure on Δ_J if θ is.

We will frequently use the following property of the operators T_J: If J_1 and J_2 are two distinct subsets of $\mathbf{N}$, then $T_{J_1 \cup J_2} = T_{J_1} \otimes T_{J_2}$.

If $T = T_{\mathbf{N}}$, and if $\mathbf{P}_\Delta$ denotes the set of probability measures on Δ, then $K = T^*(\mathbf{P}_\Delta)$ is a convex w^*-compact set of $\mathbf{P}_\Delta$. We are going to prove that for every $k \in \mathbf{N}$, every *weak* slice $S_1, \ldots, S_k$ of K, one has

$$\operatorname{diam}[(1/k)(S_1 + \cdots + S_k)] \geqq 2 \cdot ((s-2)/s) \qquad \text{for every } s \geqq 3,$$

which of course will end the proof of Theorem 4.6.

Observe first that for every $J \subset \mathbf{N}$, $|J| < \infty$, every $\sigma \in K$ is a convex combination of elements of K of the form $\rho_I^J \otimes \gamma_I$, where I runs through the subsets of J, and $\gamma_I = T_{\mathbf{N} \setminus J}^*(\theta_I)$ for some probability measure θ_I on $\Delta_{\mathbf{N} \setminus J}$. Indeed, this immediately follows from the fact that $T_J^*(\delta_I^J) = \rho_I^J$.

Let now $(\varphi_i)_{1 \leqq i \leqq k} \in \mathcal{M}(\Delta)^*$, $\| \varphi_i \| \leqq 1$, and $\delta > 0$, and let

$$S_i = S(\varphi_i, \delta) = \{\sigma \in K : \varphi_i(\sigma) > M_i - \delta\},$$

where $M_i = \sup_{\sigma \in K} \varphi_i(\sigma)$.

For every $s \geqq 3$, let $J_0 \subset \mathbf{N}_s$, $|J_0| = k2^k n$, where $n = n(s, \delta/2)$ is given by Lemma 4.8. For every $i \in [1, k]$, choose $\sigma_i \in K$ such that $\varphi_i(\sigma_i) > M_i - (\delta/2)$. By the above observation (applied to σ_i) and a convexity argument, we may find $I_i \subset J_0$ such that $\varphi_i(\rho_{I_i}^{J_0} \otimes \gamma_{i,I_i}) \geqq M_i - (\delta/2)$.

By a cardinality argument, there exists $J \subset J_0$, $|J| \geqq kn(s, \delta/2)$, which satisfies either $J \subset I_i$ or $J \subset J_0 \setminus I_i$ for every $i \in [1, k]$.

If we put $\bar{\gamma}_i = \mu_{s, I_i \setminus J}^{J_0 \setminus J} \otimes \gamma_{i,I_i}$, we have, since $J_0 \subset \mathbf{N}_s$, that $\rho_{I_i}^{J_0} \otimes \gamma_{i,I_i} = \mu_{s, J \cap I_i}^J \otimes \bar{\gamma}_i$.

Define now elements $\psi_i \in \mathcal{M}(\Delta_J)^*$ by $\psi_i(\sigma) = \varphi_i(\sigma \otimes \bar{\gamma}_i)$, and apply Lemma 4.8 to find $p \in J$ such that $\sup_{I \in \{\phi, J\}} \sup_{1 \leqq i \leqq k} \psi_i(\bar{\mu}_{s,I}^{J,p} - \bar{\bar{\mu}}_{s,I}^{J,p}) < \delta/2$.

Without loss of generality we can suppose that $J \subset I_i$ for $1 \leqq i \leqq l$, and $J \subset J_0 \setminus I_i$ for $l < i \leqq k$ (l is some integer in $[0, k]$). By a convexity argument we deduce that

$$\{\mu_{s,J}^J \otimes \bar{\gamma}_i, \mu_{s,J \setminus \{p\}}^J \otimes \bar{\gamma}_i\} \subset S_i \qquad \text{for } 1 \leqq i \leqq l$$

and

$$\{\mu_{s,\phi}^J \otimes \bar{\gamma}_i, \mu_{s,\{p\}}^J \otimes \bar{\gamma}_i\} \subset S_i \qquad \text{for } l \leqq i \leqq k.$$

Then

$$\begin{aligned}
&\operatorname{diam}\left[\frac{1}{k}(S_1 + \cdots + S_k)\right] \\
&\geqq \frac{1}{k}\left\| (\mu^J_{s,J} - \mu^J_{s,J\setminus\{p\}}) \otimes \left(\sum_{i=1}^{l} \bar{\gamma}_i \right) + (\mu^J_{s,\{p\}} - \mu^J_{s,\phi}) \otimes \left(\sum_{i=l+1}^{k} \bar{\gamma}_i \right) \right\| \\
&= \frac{s-2}{s}\left\| (\delta_0^{(p)} - \delta_1^{(p)}) \otimes \left\{ \mu^{J\setminus\{p\}}_{s,J\setminus\{p\}} \otimes \left(\sum_{i=1}^{l} \bar{\gamma}_i \right) + \mu^{J\setminus\{p\}}_{s,\phi} \otimes \left(\sum_{i=l+1}^{k} \bar{\gamma}_i \right) \right\} \right\| \\
&= 2\,\frac{s-2}{s}.
\end{aligned}$$

This concludes the proof of Theorem 4.6, since s is arbitrary. ■

COROLLARY 4.9. *Let X be a separable Banach space containing l^1, then for every $\eta > 0$, there exists a convex w*-compact subset K of the unit ball of X^* such that* $\operatorname{diam}[\frac{1}{k}(S_1 + \cdots + S_k)] \geqq 2 - \eta$ *for every weak slices $S_1, \ldots, S_k$ of K.*

Proof. A (variant of a) result of A. Pełczyński ([P], Theorem 3.4) asserts that if a separable space X contains l^1, then for every $\eta > 0$ there exists a w^*-w^*-continuous $(1+\eta)$-isomorphism from $\mathcal{M}(\Delta)$ into X^*. Corollary 4.9 is then an immediate consequence of Theorem 4.6. ■

Recall (see [G–G–M–S], III.2 and Lemma II.1) that a Banach space X has the point of continuity property (PCP) if for every bounded set C in X and $\varepsilon > 0$ there is a non-empty relatively weakly open set V of C with $\operatorname{diam}(V) < \varepsilon$ and that X is strongly regular if for every bounded C in X and $\varepsilon > 0$ there are non-empty relatively weakly open sets $V_1, \ldots, V_n$ such that $\operatorname{diam}(n^{-1}(V_1 + \cdots + V_n)) < \varepsilon$.

PROPOSITION 4.10.

(a) *If X fails* PCP *there is, for $\varepsilon > 0$, a closed set $C \subseteq X$,* $\operatorname{diam}(C) = 1$, *such that every nonempty weakly open subset V of C satisfies* $\operatorname{diam}(V) > 1 - \varepsilon$.

(b) *If X fails to be strongly regular there is, for $\varepsilon > 0$, a closed convex set C,* $\operatorname{diam}(C) = 1$, *such that* $\operatorname{diam}(n^{-1}(V_1 + \cdots + V_n)) > 1 - \varepsilon$ *for every collection $V_1, \ldots, V_n$ of non-empty weakly open sets of C.*

PROOF. (a) The proof is straightforward: Let C be a bounded set such that

$$\inf\{\operatorname{diam}(U) : U \text{ relatively weakly open in } C,\ U \neq \varnothing\} = \alpha > 0.$$

Let U_1 be a non-empty relatively weakly open subset of C of $\operatorname{diam}(U_1) <$

$\alpha/(1-\varepsilon)$. As a relatively weakly open subset U_2 of U_1 is relatively weakly open in C and therefore $\operatorname{diam}(U_2) \geqq \alpha$, if $U_2 \neq \varnothing$, we immediately obtain that $C_1 = \bar{U}_1/\operatorname{diam}(U_1)$ fulfills the requirements.

Let us note that an analogous result (with identical proof) holds for the "convex point of continuity property" (CPCP).

(b) For the case of strong regularity the proof also follows the pattern using the subsequent Lemma 4.11. We leave the details to the reader. ■

LEMMA 4.11. *Let C be a bounded subset of a Banach space X, $V_1, \ldots, V_n$ non-empty relatively weakly open subsets of C and $D = n^{-1}(V_1 + \cdots + V_n)$. Let $U_1, \ldots, U_m$ be non-empty relatively weakly open subsets of D. Then there are $n.m$ non-empty relatively weakly open subsets $W_{1,1}, \ldots, W_{m,n}$ of C such that*

$$(n.m)^{-1}(W_{1,1} + \cdots + W_{m,n}) \subseteq m^{-1}(U_1 + \cdots + U_m).$$

PROOF. Fix $1 \leqq j \leqq m$ and let $x \in U_j$. We may write $x = n^{-1}(x_1 + \cdots + x_n)$ where $x_i \in V_i$. As U_j is a relative weak neighbourhood of x in D we may find $x_1^*, \ldots, x_p^* \in X^*$ and $\delta > 0$ such that $U_j \supseteq \{y \in D : |\langle x - y, x_q^* \rangle| < \delta \text{ for } 1 \leqq q \leqq p\}$.

Define, for $1 \leqq i \leqq n$, $W_{j,i} = \{y \in V_i : |\langle y - x_i, x_q^* \rangle| < \delta \text{ for } 1 \leqq q \leqq p\}$.

Clearly $W_{j,i}$ is relatively weakly open in C and $n^{-1}(W_{j,1} + \cdots + W_{j,n}) \subseteq U_j$.

Hence, by doing the above construction for each $1 \leqq j \leqq m$, we obtain

$$(nm)^{-1} \sum_{j=1}^{m} \sum_{i=1}^{n} W_{j,i} \subseteq m^{-1} \sum_{j=1}^{m} U_j. \qquad ■$$

5. A counterexample

We give the counterexample answering negatively the problem raised in Remark 2.5.

THEOREM 5.1. *There is a constant $\beta < 2$ such that every closed convex subset C of the unit ball of the predual J_*T of JT has a slice of diameter less than or equal to β.*

REMARK 5.2. In view of the properties of JT^* one may think that $\forall \varepsilon > 0$ the statement of the above theorem holds with $\beta = \sqrt{2} + \varepsilon$. However, we only prove that Theorem 5.1 holds true for some β strictly less than 2.

We start by recalling the notations and properties of the spaces JT, its predual and its duals. We refer to [L–S] for detailed study of these spaces and to [Br] for a general account on James spaces on partially ordered sets.

Let us first recall the standard terminology on the binary tree $T = \bigcup_{n=1}^{\infty} \Delta_n$, where $\Delta_n = \{0, 1\}^n$.

An element $t \in T$ is said to be of length n, written $|t| = n$, if $t \in \Delta_n$.

We say that $t = (t_1, \ldots, t_n) \in \Delta_n$ is less (in the tree order) than $s = (s_1, \ldots, s_m) \in \Delta_m$, and write $t < s$, if $n < m$ and $t_i = s_i$ for $1 \leqq i \leqq n$.

A segment S of T is a totally ordered set $\{t_i : n \leqq i < m\}$ where $|t_i| = i$ for every $n \leqq i < m$ and $n \in \mathbf{N}$, $m \in \mathbf{N} \cup \{\infty\}$, (i.e., we also allow infinite segments). In the particular case where $n = 1$, $m = \infty$ such a segment will be called a branch. Each branch of T can be canonically identified with an element γ of $\Delta = \{0, 1\}^{\mathbf{N}}$ and conversely.

The space JT is defined as the completion of the finitely supported families $x = (x_t)_{t \in T}$ with respect to the norm

$$\| x \|_{JT} = \sup \left| \sum_{i=1}^{n} \left(\sum_{t \in S_i} x_t \right)^2 \right|^{1/2}$$

where the sup is taken over all disjoint families $\{S_i : 1 \leqq i \leqq n\}$ of segments of T. Note that we obtain the same space if we allow in the above definition the segments S_i only to be finite.

The analysis of JT^* requires one to represent its elements as functions on the set $\bar{T} = T \cup \Delta_\omega$, where Δ_ω is a copy of $\{0, 1\}^{\mathbf{N}}$. This identification is done as follows: Let $z^* \in JT^*$.

$$\text{For } t \in T, \quad z^*(t) = z^*(e_t), \quad \text{where } e_t = (\delta_{t,s})_{s \in T}.$$
$$\text{For } \gamma \in \Delta_\omega, \quad z^*(\gamma) = \lim_{t \in \gamma} z^*(t).$$

Note that the above limits exist by [L–S].

In particular $z^* \in JT^*$ defines a continuous function on $\bar{T}$, if $\bar{T}$ is equipped with "the order topology", i.e., all the points $t \in T$ are isolated points, and for $\gamma \in \Delta$, a basis of neighborhoods is $\{\{t \in \Delta : t \leqq \gamma, |t| \geqq n\}; n \geqq 1\}$.

With the above identification we define a family of operators $\pi_n : JT^* \to l^2(\Delta_n)$, $n \geqq 1$, by $\pi_n(z^*) = (z^*(t))_{t \in \Delta_n}$.

We also define $\pi_\omega(z^*) = (z^*(\gamma))_{\gamma \in \Delta_\omega}$. It is shown in [L–S] that π_ω defines a quotient map from JT^* onto $l^2(\Delta)$.

It also follows from [L–S] that the subspace B of JT^* defined by

$$B = \{z^* \in JT^* : \pi_\omega(z^*) = 0\}$$

is the (unique) isometric predual of JT and will therefore be denoted by J_*T.

One of the important tools to prove Theorem 5.1 and the main theorem of the next section is the notion of molecules.

A molecule is an element of JT^* of the form $m = \Sigma_{i=1}^n \lambda_i \chi_{S_i}$, where $S_1, \ldots, S_n$ are disjoint segments of $\bar{T}$, $\lambda_i \in \mathbf{R}$, $\Sigma_{i=1}^n \lambda_i^2 \leqq 1$. The set of molecules will be denoted by M.

To prove Theorem 5.1, we start with an easy proposition.

PROPOSITION 5.3. *Let $\pi : X \to X$ be a projection with finite dimensional kernel and C a convex, bounded subset of X. Let*

$$A = \inf\{\operatorname{diam}(S) : S \textit{ a slice of } C\} \quad \textit{and} \quad B = \inf\{\operatorname{diam}(T) : T \textit{ a slice of } \pi(C)\}.$$

Then $A \leqq B$.

PROOF. Note that $X^* = \pi^*(X^*) \oplus \pi(X)^\perp$. Let $(x_i^*)_{i=1}^n$ be a finite ε-net in the ball of $\pi(X)^\perp$ of radius $\| I - \pi \|$ and observe that for $x^* \in X^*$, $\| x^* \| = 1$ there is $1 \leqq i \leqq n$ such that

$$(*) \qquad \| x^* - (\pi^*(x^*) + x_i^*) \| < \varepsilon.$$

Fix a slice T of $\pi(C)$, say $T = T(y^*, \delta) = \{ y \in \pi(C) : y^*(y) > M_{y^*} - \delta \}$, where $y^* \in \pi^*(X^*)$, $\| y^* \| = 1$, $\delta > 0$, and $M_{y^*} = \sup_{y \in \pi(C)} y^*(y) = \sup_{x \in C} y^*(x)$.

Let S be the slice of C defined by

$$S = S(y^*, \delta) = \{x \in C : y^*(x) > M_{y^*} - \delta\} = \pi^{-1}(T) \cap C.$$

It is easy to obtain (see [B2]) a slice S_1 of C contained in S such that $\operatorname{osc}(x_i^* \mid S_1) < \varepsilon$ for every $i \in [1, n]$.

Fix $x_1, x_2 \in S_1$, $x^* \in X^*$, $\| x^* \| \leqq 1$ and find $1 \leqq i \leqq n$ such that $(*)$ holds true to estimate:

$$\begin{aligned} |x^*(x_1 - x_2)| &\leqq |\langle \pi^* x^* + x_i^*, x_1 - x_2 \rangle| + 2\varepsilon \| C \| \\ &\leqq |\langle x^*, \pi x_1 - \pi x_2 \rangle| + \varepsilon + 2\varepsilon \| C \| \\ &\leqq \operatorname{diam}(T) + \varepsilon(1 + 2 \| C \|). \end{aligned}$$

This proves the proposition since

$$\operatorname{diam}(S_1) = \sup\{x^*(x_1 - x_2) : x_1, x_2 \in S_1, \ \| x^* \| \leqq 1\}. \qquad \blacksquare$$

REMARK. Note that in the situation of Proposition 5.3 one does not have $A = B$ in general. Indeed, let $X = c_0 \oplus_1 \mathbf{R}$, $C = \operatorname{Ball}(X)$, $\pi : X \to X$ the projection onto c_0. Then $A = 0$ while $B = 2$.

We shall apply Proposition 5.3, for $n \in \mathbf{N}$, to the restriction map $\pi_{[n,\omega]} : J_* T \to J_* T$ defined by $\pi_{[n,\omega]}(x) = x\chi_{D_n}$, where $D_n = \bigcup_{m=n}^{\infty} \Delta_m \subseteq T$.

LEMMA 5.4. *For $\varepsilon > 0$ there is $\delta > 0$ such that the following holds true: For every $n \in \mathbf{N}$ and $\lambda \in l^2(\Delta_n)$, $\| \lambda \| = 1$, one has* $\operatorname{diam}(\pi_{[n,\omega]}(S)) \leqq \sqrt{2} + \varepsilon$, *for every slice S of* $\operatorname{Ball}(J_* T)$ *of the form*

$$S = \{x \in J_* T : \| x \| \leqq 1 \text{ and } (\pi_n(x), \lambda) > 1 - \delta\}.$$

PROOF. Suppose $x \in J_* T$ is a molecule m of the form

$$m = \sum_{t \in \Delta_n} \lambda(t) \chi_{S_t} \tag{1}$$

where S_t are finite segments of the tree T such that $t \in S_t$. Note that $m \in S$ and that for m and m' of the form (1), we have $\| \pi_{[n,\omega]}(m - m') \| \leqq \sqrt{2}$.

Let now x and x' be two elements of S. It is not hard to see that, for $\delta > 0$ small enough, there are y and y' which are convex combinations of molecules of the form (1) and such that $\| x - y \| < \varepsilon/2$ and $\| x' - y' \| < \varepsilon/2$ (compare the proofs of 6.5 or 6.10). It follows that

$$\| \pi_{[n,\omega]}(x - x') \| < \| \pi_{[n,\omega]}(y - y') \| + \varepsilon < \sqrt{2} + \varepsilon$$

thus proving Lemma 5.4. ■

PROOF OF THEOREM 5.1. Obtain $\delta > 0$ from Lemma 5.4 for $\varepsilon = 3/2 - \sqrt{2}$ and choose β such that $\max(3/2, 2(1 - \delta)) < \beta < 2$. We distinguish two cases:

Case 1. $\sup\{ \| \pi_\omega(z^*) \| : z^* \in \tilde{C}\} \leqq 1 - \delta$, where $\tilde{C}$ denotes the weak-star closure of C in JT^*.

Assuming that every slice of C has diameter greater than β we may apply Theorem 1.2(ii) for $\varepsilon < (\beta/2) - (1 - \delta)$ to find $D \subseteq C$ such that $\delta_2(D) > 1 - \delta$. Noting that, for $z^* \in JT^*$, $\operatorname{dist}(z^*, J_* T) = \| \pi_\omega(z^*) \|$ we arrive at a contradiction.

Case 2. $\sup\{ \| \pi_\omega(z^*) \| : z^* \in \tilde{C}\} > 1 - \delta$. Then we may find $x \in C$, $n \in \mathbf{N}$ and $\lambda \in l^2(\Delta_n)$ such that $(\pi_n(x), \lambda) > 1 - \delta$.

By Lemma 5.4 there is a slice S of $\pi_{[n,\omega]}(C)$ of diameter less than 3/2 and by Proposition 5.3 there is a slice S_1 of C of diameter less than 3/2. ■

6. A counterexample for the non-separable case

THEOREM 6.1. *There is a constant $\alpha < 1$ such that for every subset C of the unit-ball of JT^* there is an extreme point $z^{***} \in \tilde{C}$, the weak-star closure of C in JT^{***} such that* $\operatorname{dist}(z^{***}, JT^*) \leqq \alpha$.

REMARK 6.2. The computations necessary for proving the theorem are quite messy (as usual when James space is involved). Hence we don't try to obtain the best constant α in the above theorem. Again, similarly as in Remark 5.2 one may conjecture that for every $\varepsilon > 0$, $\alpha = (\sqrt{2}/2) + \varepsilon$ will work. The idea of the proof is to exploit the phenomenon described in Remark 2.11 on every branch of the tree T.

A basic tool for understanding the structure of the space JT^* is the subsequent Lemma 6.3. Roughly speaking it states that the behaviour of elements of JT^* is characterized by the behaviour of (convex combinations of) molecules; it turns out that these are relatively easy to analyse thus allowing us to obtain structural information about JT^*:

LEMMA 6.3. *Denote M to be the set of molecules in JT^*. The unit ball of JT^* is the norm-closed convex hull of M.*

PROOF. It follows from the definition of JT that M forms a norming subset of the unit-ball of JT^*. Hence the weak-star closed convex hull of M equals the unit ball of JT^*. An easy application of the Hahn–Banach theorem shows that $\bar{M}^*$, the weak-star closure of M, contains the extreme points of the unit ball of JT^*.

As JT does not contain l^1 it follows from the l^1-theorem of Odell–Rosenthal [O–R] and Haydon [H] that the unit ball of JT^* is the norm-closed convex hull of $\bar{M}^*$. To end the proof of this lemma it suffices to prove the following.

CLAIM. *Let $MI = \{\Sigma_{j=1}^{\infty} \lambda_j \chi_{S_j} : \Sigma \lambda_j^2 \leqq 1$ and S_j are disjoint segments of $\bar{T}\}$ then $MI = \bar{M}^*$. In particular M is norm dense in $\bar{M}^*$.*

PROOF. It is clear that M is norm dense in MI. Let us show that MI is w^*-closed in JT^*.

Let $(m_\alpha)_{\alpha \in I}$, $m_\alpha = \Sigma_{j=1}^{\infty} \lambda_j^{(\alpha)} \chi_{S_j^{(\alpha)}}$ be a net in MI. We may suppose that the scalars $\lambda_j^{(\alpha)}$ are such that, for every α, $|\lambda_1^{(\alpha)}| \geqq |\lambda_2^{(\alpha)}| \geqq \cdots \geqq |\lambda_j^{(\alpha)}| \geqq \cdots$. Passing to a refinement of the net $(m_\alpha)_{\alpha \in I}$, we may also suppose that for every $j \in \mathbf{N}$, there exists a *segment* S_j of $\bar{T}$ such that $\chi_{S_j}(t) = \lim_\alpha \chi_{S_j^{(\alpha)}}(t)$ for every $t \in T$. This means that $\chi_{S_j} = \omega^* - \lim_\alpha \chi_{S_j^{(\alpha)}}$ for every $j \in \mathbf{N}$.

If $\lambda_j = \lim_\alpha \lambda_j^{(\alpha)}$, then $\Sigma \lambda_j^2 \leqq 1$ (the unit ball of l^2 is weakly compact) and hence it follows easily that $m = \Sigma \lambda_j \chi_{S_j} = \omega^* - \lim_\alpha m_\alpha$. ■

To prove Theorem 6.1 we need a closer investigation of JT^{***}.

Let $z^{***} \in JT^{***}$, and $(z_\alpha^*)_{\alpha \in I}$ a net in JT^* converging weak-star to z^{***}. As the restriction of z^{***} to JT induces an element of JT^* it follows from the analysis in [L–S] or from Lemma 6.3 above that the function z^{***} on the tree T defined by $z^{***}(t) = \lim_\alpha z_\alpha^*(t)$ converges along every infinite branch of T.

Let $\Delta_{\omega-1}$ be another copy of $\{0, 1\}^{\mathbf{N}}$, where we identify the branches of T with the points of $\Delta_{\omega-1}$. Define for $\gamma \in \Delta_{\omega-1}$, $z^{***}(\gamma) = \lim_{t \in \gamma} z^{***}(t)$. (This limit exists by the above observation.)

On the other hand the function z^{***} on Δ_ω defined by $z^{***}(\gamma) = \lim_\alpha z_\alpha^*(\gamma)$ for every $\gamma \in \Delta_\omega$ exists, since $(\pi_\omega(z_\alpha^*))_{\alpha \in I}$ is weakly Cauchy in $l^2(\Delta)$.

So we can represent z^{***} as a function on $\bar{\bar{T}} = T \cup \Delta_{\omega-1} \cup \Delta_\omega$, such that $(z^{***}(t))_{t \in \gamma}$ converges to $z^{***}(\gamma)$ for every $\gamma \in \Delta_{\omega-1}$ which we identify with an infinite branch of T, i.e., z^{***} is a continuous function on $\bar{\bar{T}}$ if we consider $\bar{\bar{T}}$ as a topological space such that $T \cup \Delta_{\omega-1}$ is "the order compactification" of T described in Section 5 and Δ_ω is a clopen subset of $\bar{\bar{T}}$. (This explains why we have chosen the somewhat unorthodox notation $\Delta_{\omega-1}$ to indicate that $\Delta_{\omega-1}$ squeezes in between T and Δ_ω.)

The operator $\pi_\omega^{**}: JT^{***} \to l^2(\Delta)$ is still given by the restriction of $z^{***} \in JT^{***}$ to Δ_ω, i.e., $\pi_\omega(z^{***}) = (z^{***}(\gamma))_{\gamma \in \Delta_\omega}$, and will therefore still be denoted by π_ω. We may also define $\pi_{\omega-1}: JT^{***} \to l^2(\Delta)$ to be the "restriction" of $z^{***} \in JT^{***}$ to $\Delta_{\omega-1}$, i.e., $\pi_{\omega-1}(z^{***}) = (z^{***}(\gamma))_{\gamma \in \Delta_{\omega-1}}$.

Observe that if i denotes the natural injection from JT^* into JT^{***}, then $i(JT^*) = \ker(\pi_\omega - \pi_{\omega-1})$.

Observe also that $(\pi_\omega - \pi_{\omega-1})$ is an operator onto $l^2(\Delta)$, hence $(\pi_\omega - \pi_{\omega-1})$ induces an isomorphism between $l^2(\Delta)$ and JT^{***}/JT^*. In particular, for every $z^{***} \in JT^{***}$ one has $\operatorname{dist}(z^{***}, i(JT^*)) \leqq C \| (\pi_\omega - \pi_{\omega-1})z^{***} \|$ for some constant C. The main step in proving Theorem 6.1 is to show that $\operatorname{dist}(z^{***}, i(JT^*)) = \| (\pi_\omega - \pi_{\omega-1})z^{***} \| / \sqrt{2}$.

Let us now give some technical lemmas on the structure of the molecules and the operators π_ω and $\pi_{\omega-1}$.

Lemma 6.4. *Let $z^{***} \in \tilde{M} = \bar{M}^{\sigma(JT^{***}, JT^{**})}$ where M denotes the set of molecules of JT^*, $\xi = \pi_\omega(z^{***})$, and $\eta = \pi_{\omega-1}(z^{***})$.*

There is $\zeta \in l^2(\Delta)$, and a set $A \subset \Delta$, $A \subseteq \operatorname{supp}(\xi)$, $A \cap \operatorname{supp}(\zeta) = \varnothing$ such that $\eta = \xi \cdot \chi_A + \zeta$ and $\| \xi \|^2 + \| \zeta \|^2 \leqq 1$. Hence $\| \xi - \eta \| \leqq \sqrt{2}$.

PROOF. Let $(m_\alpha)_{\alpha\in I}$ be a net in M converging weak-star to z^{***} and write

$$m_\alpha = \sum_{j\in J(\alpha)} \lambda_j^\alpha \chi_{S_j^\alpha} = \sum_{j\in J_1(\alpha)} \lambda_j^\alpha \chi_{S_j^\alpha} + \sum_{j\in J_2(\alpha)} \lambda_j^\alpha \chi_{S_j^\alpha} = m_\alpha^1 + m_\alpha^2 .$$

Here S_j^α are segments of $\bar{T}$, $J(\alpha)$ finite index sets and

$$J_1(\alpha) = \{ j \in J(\alpha) : S_j^\alpha \cap \operatorname{supp}(\xi) \neq \varnothing \} \quad \text{and} \quad J_2(\alpha) = J(\alpha) \setminus J_1(\alpha)$$

where we consider ξ as a function on Δ. Clearly for every $\gamma \in \Delta$, $\gamma \in \operatorname{supp}(\xi)$, $(m_\alpha^1(\gamma))_{\alpha\in I}$ converges to $\xi(\gamma)$. Hence $\lim_{\alpha\in I} \Sigma_{j\in J_2(\alpha)} |\lambda_j^\alpha|^2 \leqq 1 - \|\xi\|^2$.

Let x^{***} and y^{***} be weak-star cluster points in JT^{***} of $(m_\alpha^1)_{\alpha\in I}$ and $(m_\alpha^2)_{\alpha\in I}$ respectively and let $\zeta = \pi_{\omega-1}(y^{***})$, which clearly satisfy $\|\xi\|^2 + \|\zeta\|^2 \leqq 1$. For $\gamma \in \Delta$ distinguish two cases:

Case 1. There exists a segment S of $\bar{T}$ containing γ such that, for α big enough, there is $j \in J_1(\alpha)$ with $S_j^\alpha \supseteq S$. In this case we get $\xi(\gamma) = \pi_\omega(z^{***})(\gamma) = \pi_{\omega-1}(x^{***})(\gamma) = \eta(\gamma)$.

Case 2. If case 1 does not happen, we get $\pi_{\omega-1}(x^{***})(\gamma) = 0$ hence $\pi_{\omega-1}(z^{***})(\gamma) = \eta(\gamma) = \pi_{\omega-1}(y^{***})(\gamma) = \zeta(\gamma)$.

Let A be the set of the points in Δ such that case 1 happens. Then $\eta = \xi \cdot \chi_A + \zeta$. Finally observe that

$$(*) \quad \|\xi - \eta\| \leqq \|\xi - \xi\cdot\chi_A\| + \|\eta - \xi\cdot\chi_A\| \leqq \|\xi\| + \|\zeta\| \leqq \sqrt{2}.$$

PROPOSITION 6.5. *For every $\varepsilon > 0$, there exists $\delta > 0$, such that for every $z^{***} \in JT^{***}$, $\|z^{***}\| \leqq 1$, we have the following*:

$$(**) \quad \|\xi - \eta\| > \sqrt{2}(1-\delta) \Rightarrow \|\xi + \eta\| \leqq \varepsilon \text{ and } \|\xi\|^2 + \|\eta\|^2 \geqq 1 - \varepsilon$$

where $\xi = \pi_\omega(z^{***})$, and $\eta = \pi_{\omega-1}(z^{***})$.

PROOF. From $(*)$ and using the relation $\|\xi\|^2 + \|\zeta\|^2 \leqq 1$ one easily checks that for every $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that $(**)$ holds for every $z^{***} \in \tilde{M}$. [To do so check the following estimates: $\|\xi\| \sim (\sqrt{2}/2)$; $\|\xi - \xi\chi_A\| \sim (\sqrt{2}/2)$; $\|\xi\chi_A\| \sim 0$; $\|\eta\| \sim (\sqrt{2}/2)$; $\|\xi + \eta\| \sim 0$.]

Fix now $\varepsilon > 0$, and let $z^{***} \in JT^{***}$, $\|z^{***}\| \leqq 1$ such that $\|\xi - \eta\| > \sqrt{2}(1-\delta)$ (the value of δ will be fixed later). Since $l^1 \not\subset JT^{**}$, the l^1-theorem of Odell–Rosenthal [O–R] and Haydon [H] implies that $\overline{\operatorname{conv}}\ \tilde{M} = B(JT^{***})$. So we can find $n \in \mathbf{N}$, and elements $(z_i^{***})_{i\leqq n} \in \tilde{M}$ such that

$$\left\| z^{***} - \frac{1}{n}\sum_{i=1}^n z_i^{***} \right\| \leqq \delta\sqrt{2}.$$

Let $\xi_i = \pi_\omega(z_i^{***})$ and $\eta_i = \pi_{\omega-1}(z_i^{***})$, and

$$I = \{i \in [1, n] : \| \xi_i - \eta_i \| \leqq \sqrt{2}(1 - \sqrt{\delta})\}.$$

Then it is not difficult to see that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (\xi_i - \eta_i) \right\| \geqq \sqrt{2}(1 - 3\delta),$$

and $|I| \leqq 3n\sqrt{\delta}$.

If we assume that $\sqrt{\delta} < \delta_1(\varepsilon/2)$, we deduce that

$$\| \xi + \eta \| < 2\delta\sqrt{2} + 6\sqrt{\delta} + \varepsilon/2 < \varepsilon$$

if δ is small enough.

On the other hand we have

$$\| \xi \|^2 + \| \eta \|^2 = \tfrac{1}{2}(\| \xi - \eta \|^2 + \| \xi + \eta \|^2) > (1 - \delta)^2 > 1 - \varepsilon$$

if $\delta \leqq \varepsilon/2$. ∎

PROPOSITION 6.6. *For $z^{***} \in JT^{***}$ we have*

$$\| (\pi_\omega - \pi_{\omega-1})(z^{***}) \| \leqq \sqrt{2} \| z^{***} \|.$$

PROOF. The function $d : JT^{***} \to \mathbf{R}_+$ defined by $d(z^{***}) = \| (\pi_\omega - \pi_{\omega-1})(z^{***}) \|$ is convex and norm-continuous. As the convex hull of $\tilde{M}$ is norm-dense in the unit-ball of JT^{***}, we infer from Lemma 6.4 that $\| (\pi_\omega - \pi_{\omega-1})(z^{***}) \| \leqq \sqrt{2}$ if $\| z^{***} \| \leqq 1$. ∎

PROPOSITION 6.7. *Denote $i : JT^* \to JT^{***}$ the canonical injection from JT^* into its double dual. Then* $\operatorname{dist}(z^{***}, i(JT^*)) = \| (\pi_\omega - \pi_{\omega-1})(z^{***}) \| / \sqrt{2}$, *for every $z^{***} \in JT^{***}$.*

PROOF. Let $\xi = \pi_\omega(z^{***})$ and $\eta = \pi_{\omega-1}(z^{***})$. Find $z^* \in JT^*$ such that $\pi_\omega(z^*) = (\xi + \eta)/2$; then for $y^{***} = z^{***} - i(z^*)$ we obtain

$$\pi_\omega(y^{***}) = \xi - (\xi + \eta)/2 = (\xi - \eta)/2$$

and

$$\pi_{\omega-1}(y^{***}) = \eta - (\xi + \eta)/2 = (\eta - \xi)/2,$$

and clearly $\operatorname{dist}(z^{***}, i(JT^*)) = \operatorname{dist}(y^{***}, i(JT^*))$.

Hence there is no loss of generality in assuming in the statement of the proposition that $\xi = -\eta$.

Fix $\varepsilon > 0$, and let $A = \{a_1, \ldots, a_k\}$ be a finite subset of Δ such that

$\| \xi - \xi\chi_A \| < \varepsilon$. Choose now disjoint segments $S_1, \ldots, S_k$ of $\bar{T}$ ending at $a_1, \ldots, a_k$ respectively.

Then the molecule $m = \Sigma_{j=1}^k \xi(a_j)\chi_{S_j}$ satisfies $\| m \| = (\Sigma_{j=1}^k |\xi(a_j)|^2)^{1/2} \leqq \| \xi \|$, and $\| \xi - \pi_\omega(m) \| < \varepsilon$.

We may assume that there is $n \in \mathbf{N}$ such that all S_j start at level n. For $p > n$ and $1 \leqq j \leqq k$, let $S_j^p = S_j \cap (\Delta_n \cup \Delta_{n+1} \cup \cdots \cup \Delta_p)$, $T_j^p = S_j \setminus S_j^p$, and $x_p^* = \Sigma_{j=1}^k \xi(a_j)(-\chi_{S_j^p} + \chi_{T_j^p})$.

Then $\| x_p^* \| \leqq \sqrt{2} \| \xi \|$ and $(i(x_p^*))_{p=1}^\infty$ converges weak-star to an element $x^{***} \in JT^{***}$ such that $\| \pi_\omega(x^{***}) - \xi \| < \varepsilon$ and $\| \pi_{\omega-1}(x^{***}) + \xi \| = \| \pi_{\omega-1}(x^{***}) - \eta \| < \varepsilon$. Hence

$$\begin{aligned}\operatorname{dist}(z^{***}, i(JT^*)) &\leqq \operatorname{dist}(x^{***}, i(JT^*)) + \operatorname{dist}(z^{***} - x^{***}, i(JT^*)) \\ &\leqq \| x^{***} \| + C \| (\pi_\omega - \pi_{\omega-1})(z^{***} - x^{***}) \| \\ &\leqq \sqrt{2} \| \xi \| + 2C\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary we have proved the implication "$\leqq$".

As regards the reverse inequality observe that

$$\begin{aligned}\sqrt{2} \cdot \operatorname{dist}(z^{***}, i(JT^*)) &= \sqrt{2} \cdot \inf\{ \| z^{***} - z^* \| : z^* \in JT^* \} \\ &\geqq \inf\{ \| (\pi_\omega - \pi_{\omega-1})(z^{***} - z^*) : z^* \in JT^* \} \\ &= \| (\pi_\omega - \pi_{\omega-1})(z^{***}) \| .\end{aligned}$$ ■

An immediate corollary of Propositions 6.5 and 6.7 is the following:

COROLLARY 6.8. *For $\varepsilon > 0$ there is $\delta > 0$ such that, for $z^{***} \in JT^{***}$, $\| z^{***} \| = 1$, with* $\operatorname{dist}(z^{***}, i(JT^*)) > 1 - \delta$, *one has*

$$\| \pi_\omega(z^{***}) + \pi_{\omega-1}(z^{***}) \| < \varepsilon \text{ and } \| \pi_\omega(z^{***}) \|^2 + \| \pi_{\omega-1}(z^{***}) \|^2 > 1 - \varepsilon.$$

For the next lemma we need to introduce some notations.

Denote $p_n : \Delta \to \Delta_n$ the natural projection, defined by $p_n(\{\varepsilon_i\}_{i=1}^\infty) = \{\varepsilon_i\}_{i=1}^n$. If A is a finite subset of Δ such that p_n, restricted to A, is one to one and $f \in l^2(\Delta)$ with $\operatorname{supp}\{f\} \subset A$, we define the element $p_n(f)$ of $l^2(\Delta_n)$ by

$$p_n(f)(t) = \begin{cases} f(\gamma) & \text{if } t \leqq \gamma \text{ for some } \gamma \in A, \\ 0 & \text{if not.} \end{cases}$$

With this notation we formulate an immediate consequence of 6.8.

LEMMA 6.9. *For $\varepsilon > 0$ there is $\delta > 0$ such that, for $z^{***} \in JT^{***}$,*

$\| z^{***} \| = 1$ and $\mathrm{dist}(z^{***}, i(JT^*)) > 1 - \delta$, for every net $(z_\alpha^*)_{\alpha \in I}$ in $i(JT^*)$ converging to z^{***} and $n \in \mathbf{N}$ there is $\xi \in l^2(\Delta)$, $\| \xi \| = 1$, with finite support A and $n \leqq r < s$, $r, s \in \mathbf{N}$ such that p_r is one to one on A and such that there is $\alpha \in I$ with

(i) $(p_s(\xi), \pi_s(z_\alpha^*))_{l^2(\Delta_s)} > \sqrt{2}/2 - \varepsilon$

and

(ii) $(p_r(\xi), \pi_r(z_\alpha^*))_{l^2(\Delta_r)} < -\sqrt{2}/2 + \varepsilon$.

Proof. Fix $\varepsilon > 0$. By 6.8 there is $\delta > 0$ such that $\mathrm{dist}(z^{***}, i(JT^*)) > 1 - \delta$ implies that there is $\xi \in l^2(\Delta)$ with finite support, $\| \xi \| = 1$ and such that

$$\| \pi_\omega(z^{***}) - \xi/\sqrt{2} \| < \varepsilon \quad \text{and} \quad \| \pi_{\omega - 1}(z^{***}) + \xi/\sqrt{2} \| < \varepsilon.$$

Using the definition of $z^{***}(\gamma)$ for $\gamma \in \Delta_{\omega - 1}$, we deduce that there exists $r \geqq n$, such that p_r is one to one on the support of ξ, and such that $(\pi_r(z^{***}), p_r(\xi)) < -\sqrt{2}/2 + \varepsilon$, hence for $\alpha \geqq \alpha_0(r)$, we have $(\pi_r(z_\alpha^*), p_r(\xi)) < -\sqrt{2}/2 + \varepsilon$.

On the other hand for $\alpha \geqq \alpha_1 \geqq \alpha_0(r)$, we have $(\pi_\omega(z_\alpha^*), \xi) > \sqrt{2}/2 - \varepsilon$. So there exists $s > r$ such that $(\pi_s(z_\alpha^*), \xi) > \sqrt{2}/2 - \varepsilon$ (since the elements of JT^* are continuous functions on $\bar{T}$). ■

We now establish the final technical tool for the proof of Theorem 6.1:

Lemma 6.10. *There is an absolute constant $\kappa > 0$ such that the following situation can not occur:*

There exists $\xi, \lambda \in l^2(\Delta)$, $\| \lambda \| = \| \xi \| = 1$ with finite supports, $z^* \in JT^*$, $\| z^* \| = 1$, and $n \leqq r < s \leqq m$, such that:

(0) *p_n is an injection both on $A = \mathrm{supp}(\xi)$ and $B = \mathrm{supp}(\lambda)$,*

(i) $(p_s(\xi), \pi_s(z^*))_{l^2 \in (\Delta_s)} > \sqrt{2}/2 - \kappa$,

(ii) $(p_r(\xi), \pi_r(z^*))_{l^2(\Delta_r)} < -\sqrt{2}/2 + \kappa$,

(iii) $\frac{1}{2}[(p_n(\lambda), \pi_n(z^*))_{l^2(\Delta_n)} + (p_m(\lambda), \pi_m(z^*))_{l^2(\Delta_m)}] > \frac{2}{3}$.

Proof. Suppose first that z^* is a molecule of the form

$$(1) \qquad z^* = \left[\sum_{t \in p_r(A)} - p_r(\xi)(t) \cdot \chi_{S_t^1} + \sum_{t \in p_s(A)} p_s(\xi)(t) \cdot \chi_{S_t^2} \right] \Big/ \sqrt{2}$$

where $\{(S_t^1)_{t \in p_r(A)}, (S_t^2)_{t \in p_s(A)}\}$ is a disjoint family of segments in T such that t is an element of S_t^1 (resp. of S_t^2), for every $t \in p_r(A)$ (resp. every $t \in p_s(A)$). Note that such a z^* satisfies (i) and (ii) for each $\kappa > 0$.

Let $B_1 = (B \setminus A) \cup \{t \in A \cup B$ such that $\mathrm{sign}(\xi) = \mathrm{sign}(\lambda)\}$ and $B_2 = B \setminus B_1 = \{t \in A \cap B$ such that $\mathrm{sign}(\xi) \neq \mathrm{sign}(\lambda)\}$ and let $\lambda_1 = \lambda \chi_{B_1}$ and $\lambda_2 = \lambda \chi_{B_2}$.

Note that $(p_n(\lambda_1), \pi_n(z^*)) \leqq 0$ as none of the segments S_t^2 can pass through $p_n(B_1)$ and the contribution from the first term in (1) can only be less than or equal to zero.

Similarly we get $(p_m(\lambda_2), \pi_m(z^*)) \leqq 0$ as none of the segments S_t^1 can pass through $p_m(B_2)$.

Hence

$$\begin{aligned} &\tfrac{1}{2}[(p_n(\lambda), \pi_n(z^*)) + (p_m(\lambda), \pi_m(z^*))] \\ &\quad \leqq \tfrac{1}{2}[(p_n(\lambda_2), \pi_n(z^*)) + (p_m(\lambda_1), \pi_m(z^*))] \\ &\quad \leqq \tfrac{1}{2}[\|\lambda_2\| \cdot \|\pi_n(z^*)\| + \|\lambda_1\| \cdot \|\pi_m(z^*)\|] \\ &\quad = (\tfrac{1}{2}\sqrt{2})[\|\lambda_2\| + \|\lambda_1\|] \leqq \tfrac{1}{2}, \end{aligned}$$

a contradiction to (iii).

If $z^* = \Sigma_{i \in I} \mu_i \cdot z_i^*$ is a convex combination of molecules, where I is a finite set, z^* satisfying (i), (ii) and (iii) for $\kappa = 10^{-7}$, there is $I_1 \subseteq I$ such that $\Sigma_{i \in I \setminus I_1} \mu_i < 10^{-6}$ and for $i \in I_1$, z_i^* satisfies (i) and (ii) for $\kappa = 10^{-6}$.

One easily verifies that, for $i \in I_1$, there is a molecule y_i^* of the form as given in (1) such that $\| z_i^* - y_i^* \|_{JT^*} < 10^{-2}$.

Hence there is a convex combination $y^* = \Sigma_{i \in I_1} \kappa_i \cdot y_i^*$ where y_i^* are molecules of the form described in (1) and such that $\| z^* - y^* \|_{JT^*} < 10^{-1}$.

As the expression given by (iii) gives a value less than or equal to $\frac{1}{2}$ for y^* (by the first part of the proof) we arrive at the desired contradiction (obtaining 10^{-7} as a possible choice for κ). ∎

PROOF OF THEOREM 6.1. We distinguish two cases:

Case 1. $\sup\{ \| \pi_\omega(z^*) \| : z^* \in C \} \leqq 2/3$. Then (since $\pi_\omega(\tilde{C}) = \pi_\omega(\widetilde{C})$) it is easy to obtain from Corollary 6.8 an absolute constant $\alpha_1 < 1$ such that $\operatorname{dist}(z^{***}, i(JT^*)) < \alpha_1$ for every z^{***} of $\tilde{C}$.

Case 2. There is $z^* \in C$ such that $\| \pi_\omega(z^*) \| > 2/3$.

Let $\kappa > 0$ be given by 6.10 and determine $\delta > 0$ from 6.9 by taking $\varepsilon = \kappa$. Let $\alpha_2 = 1 - \delta$.

We may find $\lambda \in l^2(\Delta)$, $\| \lambda \| = 1$ of finite support B and $n \in \mathbf{N}$ such that p_n is injective on B and such that $\frac{1}{2}[(p_n(\lambda), \pi_n(z^*)) + (\lambda, \pi_\omega(z^*))] > 2/3$.

Define a slice S of C by $S = \{x^* \in C : \frac{1}{2}[(p_n(\lambda), \pi_n(x^*)) + (\lambda, \pi_\omega(x^*))] > 2/3\}$, which is non-empty as $z^* \in S$.

The weak-star closure $\tilde{S}$ of S in JT^{***} contains an extreme point z_0^{***} of $\tilde{C}$. If $\operatorname{dist}(z_0^{***}, i(JT^*))$ is bigger than α_2, we derive from 6.9 that there is $z_0^* \in S$ and $\xi \in l^2(\Delta)$ and $n \leqq r < s$ such that (i) and (ii) of 6.9 are satisfied with $\varepsilon = \kappa$.

On the other hand $z_0^* \in S$, hence there is $m \in \mathbf{N}$, $s < m$ such that

$$\tfrac{1}{2}[(p_n(\lambda), \pi_n(z_0^*)) + (p_m(\lambda), \pi_m(z_0^*))] > 2/3.$$

Hence we are exactly in the situation which is proven to be contradictory by 6.10.

Finally let $\alpha = \max(\alpha_1, \alpha_2)$ to finish the proof. ■

7. Another counterexample

In Section 2 we gave a geometrical proof of Theorem 1.1(iii), and the previous example shows that the separability assumption in this theorem cannot be dropped.

One may also give a more analytical proof of 1.1(iii). We are going to show in this section that separability is also necessary in a crucial step of this analytical proof. The counterexample of this section also answers negatively a question of K. Musiał (see Remark 7.3).

We start by giving the analytic proof of Theorem 1.1(iii).

Recall that the extreme points K of $\tilde{F}$ equipped with the w^*-topology (i.e., the topology $\sigma(L^1(\Delta)^{**}, L^1(\Delta)^*)$) is a compact space called the Stone space of $L^\infty(\Delta)$. The space $L^\infty(\Delta)$ may naturally be identified with $C(K)$ and we shall write $\hat{f}$ for $f \in L^\infty(\Delta)$ if we regard f as a continuous function on K. (For details we refer to [T], 1.4.)

If we denote by μ the Haar measure on Δ, there is a unique Radon-measure $\hat{\mu}$ on K such that for every $f \in L^\infty(\Delta)$ the equality $\int_\Delta f(\gamma)d\mu(\gamma) = \int_K \hat{f}(s)d\hat{\mu}(s)$ holds true.

We shall need the following fact about the Stone space $(K, \hat{\mu})$, which follows immediately from [T], 1.4.3a: For every $\hat{\mu}$-measurable set $\Omega \subseteq K$, $\hat{\mu}(\Omega) > 0$, and every $\varepsilon > 0$ there exists an μ-measurable $B \subseteq \Delta$, $\mu(B) > \hat{\mu}(\Omega) - \varepsilon$, such that $\Omega \supset \hat{B}$, where $\hat{B}$ is the clopen subset of K corresponding to B, i.e., $\hat{B} = \mathrm{Ext}(\tilde{F}_B)$.

LEMMA 7.1. *Let Y be a separable Banach space and $T: L^1(\Delta) \to Y$ be a continuous linear operator. Let φ denote the restriction to K of the function $d_Y \circ T^{**}: L^1(\Delta)^{**} \to \mathbf{R}_+$, where $d_Y(y^{**}) = \mathrm{dist}(y^{**}, Y)$, The function φ is $\hat{\mu}$-measurable. Moreover, T is representable iff φ equals zero $\hat{\mu}$-almost everywhere.*

PROOF. For $y \in Y$, $y^{**} \in Y^{**}$, let $d_y(y^{**}) = \| y^{**} - y \|$.

Note that d_y is a lower semicontinuous function on (Y^{**}, w^*), hence $d_y \circ T^{**}$ is lower semicontinuous and therefore $\hat{\mu}$-measurable.

Let $(y_n)_{n=1}^{\infty}$ be a dense sequence in Y and note that $d_Y \circ T^{**} = \inf_{n\in\mathbf{N}} d_{y_n} \circ T^{**}$, which shows that φ is $\hat{\mu}$-measurable.

As regards the last assertion it follows from the remark preceding the lemma that $\hat{\mu}(\{\varphi = 0\}) = 1$ iff, for every $\varepsilon > 0$, there is $B \subseteq \Delta$, $\mu(B) > 1 - \varepsilon$, such that $\hat{B} \subseteq \{\varphi = 0\}$.

The latter assertion is equivalent to saying that $T^{**}(\tilde{\mathscr{F}}_B) \subseteq Y$, i.e., that $T \circ R_B$ is weakly compact, where $R_B : L^1(\Delta) \to L^1(\Delta)$ is the multiplication with the indicator function of B.

As is well known ([D–U], Prop. III.2.21) this "almost weak compactness" is equivalent to the representability of T. The lemma is proved. ∎

After these preliminaries we can give an alternative proof of Theorem 1.1(iii):

If Y fails RNP there is a non-representable operator $T : L^1(\Delta) \to Y$. Hence $\| \varphi \|_{L^\infty(K,\hat{\mu})} \neq 0$ and we may assume $\| \varphi \|_{L^\infty(K,\hat{\mu})} = 1 - \varepsilon/2$.

Find a $\hat{\mu}$-measurable set $\Omega \subseteq K$, $\hat{\mu}(\Omega) > 0$, such that φ is bigger than $1 - \varepsilon$ on Ω. Let $(y_n)_{n=1}^{\infty}$ be dense in Y and denote φ_n the restriction of $d_{y_n} \circ T^{**}$ to K. As $\varphi = \inf\{\varphi_n : n \in \mathbf{N}\}$ we may find $n_0 \in \mathbf{N}$ and an $\hat{\mu}$-measurable subset Ω_1 of Ω, $\hat{\mu}(\Omega_1) > 0$, such that φ_{n_0} is smaller than 1 on Ω_1. Let B be a μ-measurable subset of Δ, $\mu(B) > 0$, such that $\hat{B} \subseteq \Omega_1$.

Note that $T^{**}(\tilde{\mathscr{F}}_B)$ is contained in the ball of radius 1 around y_{n_0}. As every extreme point of $\widetilde{T(\mathscr{F}_B)} = T^{**}(\tilde{\mathscr{F}}_B)$ is the image under T^{**} of an extreme point of $\tilde{\mathscr{F}}_B$, we conclude that every extreme point of $\widetilde{T(\mathscr{F}_B)}$ has distance from Y bigger than $1 - \varepsilon$.

Finally, letting $C =$ closed convex hull$\{(T(\mathscr{F}_B) - y_{n_0}), -(T(\mathscr{F}_B) - y_{n_0})\}$ we have constructed again a set satisfying the requirements of Theorem 1.1(iii). ∎

The separability is necessary to insure the measurability of the function φ:

THEOREM 7.2. *There exists a non-separable space X and an operator $S : L^1(\Delta) \to X$ such that φ is not measurable.*

We first fix some notation.

Let A be a subset of Δ. We define the subspace Z_A of JT^* as

$$Z_A = \{x^* \in JT^* : \pi_\omega(x^*)(\gamma) = 0 \text{ for all } \gamma \in A\}.$$

One quickly verifies that Z_A is a closed subspace of JT^*. Note that $JT^* = Z_\phi$ and $J_* T = B = Z_\Delta$.

Define $F : \Delta \to JT^*$ by $F(\gamma) = \chi_{b(\gamma)}$, where $b(\gamma)$ is the branch of $\bar{T}$ starting at the origin of T and ending at $\gamma \in \Delta_\omega$ and $\chi_{b(\gamma)}$ denotes the indicator function of this branch. Clearly F is a weak-star scalarly measurable function (in fact, F is weakly scalarly measurable) hence we may define $T: L^1(\Delta, \mu) \to JT^*$ by

$$T(f) = w^* - \int_\Delta f(\gamma) \cdot F(\gamma) d\mu(\gamma).$$

One easily checks that T defines in fact an operator into B — still denoted by T — and that the integral even makes sense as a Pettis integral. In fact, T is the "canonical" non-representable operator from L^1 to B.

PROOF OF THEOREM 7.2. We are going to show that if $A \subset \Delta$ is of μ-inner measure 0 and outer measure 1, then the statement of Theorem 7.2 holds with $X = Z_A$ and $S = T_A = j_A \circ T$, where $j_A : B \to Z_A$ denotes the canonical embedding.

We first have to identify the double-dual of Z_A:

Let $i_A : Z_A \to JT^*$ be the canonical embedding. Using the notation of Sections 5 and 6, the space Z_A^{**} — strictly speaking $i_A^{**}(Z_A^{**})$ — is the subspace of JT^{***} given by

$$i_A^{**}(Z_A^{**}) = \{x^{***} \in JT^{***} : \pi_\omega(x^{***})(\gamma) = 0 \text{ for all } \gamma \in A\}.$$

We shall identify $Z_A{}^{**}$ with a subspace of JT^{***} which in turn we represent as a space of functions on $\bar{\bar{T}}$.

We now investigate φ: As in 2.8(2) let ψ denote the canonical surjection of the Stone space K onto Δ obtained by restricting the elements of K, which is a subset of $L^\infty(\Delta, \mu)^*$, to the subspace $C(\Delta)$ of $L^\infty(\Delta, \mu)$ and identifying Δ with the Dirac measures in $C(\Delta)^*$.

Note that the operator $(T_A)^* : Z_A^* \to L^\infty(\Delta, \mu)$ takes its values in $C(\Delta)$. Indeed it suffices to verify that $T^* : B^* \to L^\infty(\Delta, \mu)$ takes its values in $C(\Delta)$ which is obviously true, as in $B^* = JT$ the unit vectors $(e_t)_{t \in T}$ span a dense subspace of JT and $T^*(e_t) = \chi_{I_t} \in C(\Delta)$, where I_t denotes the sets of elements of Δ which are successors of t (χ_{I_t} is continuous since I_t is a clopen subset of Δ).

Hence $T_A^{**} : L^\infty(\Delta, \mu)^* \to Z_A^{**}$ factors through $C(\Delta)^*$ and the restriction of T_A^{**} to K factors via $\psi : K \to \Delta$ through Δ, i.e., $T_{A|K}^{**} = S_A \circ \psi$, where S_A is a function from Δ to Z_A^{**}.

One easily verifies that, for $\gamma \in \Delta$, $S_A(\gamma) = \chi_{k(\gamma)}$ where $k(\gamma)$ is the branch in $\bar{\bar{T}} = T \cup \Delta_{\omega-1} \cup \Delta_\omega$ starting at the origin and ending at $\gamma \in \Delta_{\omega-1}$ (not containing the $\gamma \in \Delta_\omega$!) and $\chi_{k(\gamma)}$ is the indicator function of this branch. Indeed, it suffices to consider, for $\gamma \in \Delta$, a sequence $(f_n)_{n=1}^\infty$ in the positive face $\mathscr{F}$ of $L^1(\Delta, \mu)$ converging $\sigma(C(\Delta)^*, C(\Delta))$ to the Dirac measure at the point γ

and to note that $(T_A f_n)_{n=1}^{\infty}$ is a sequence in $j_A(B)$ converging pointwise on $\bar{\bar{T}}$ towards $\chi_{k(\gamma)}$.

Now make the crucial observation, similarly as in Remark 2.11 above:

If $\gamma \in A$ then similarly as in 2.11, $\operatorname{dist}(S_A(\gamma), Z_A) = 1$, while for $\gamma \notin A$ we have $\operatorname{dist}(S_A(\gamma), Z_A) \leqq \sqrt{2}/2$. Indeed, in the latter case the element $(\frac{1}{2})\chi_{\lambda(\gamma)}$, where $\lambda(\gamma)$ consists of $k(\gamma)$ plus the element γ of Δ_ω, is in Z_A — more precisely in the canonical image of Z_A in Z_A^{**} — and therefore

$$\operatorname{dist}(S_A(\gamma), Z_A) \leqq \| \chi_{k(\gamma)} - (\tfrac{1}{2})\chi_{\lambda(\gamma)} \|_{JT^{***}}.$$

Define, for $n \in \mathbf{N}$, the element $\rho_n(\gamma)$ of JT^{***} by

$$\rho_n(\gamma) = \begin{cases} \frac{1}{2} & \text{for } t \in \lambda(\gamma), \quad |t| \leqq n, \\ -\frac{1}{2} & \text{for } t \in \lambda(\gamma), \quad |t| > n \text{ (including } |t| = \omega - 1 \text{ and } |t| = \omega), \\ 0 & \text{elsewhere.} \end{cases}$$

If we denote by i the canonical embedding of JT^* into JT^{***}, then $\| \rho_n(\gamma) \| = \sqrt{2}/2$, $\rho_n(\gamma) \in i(JT^*)$, and $(\rho_n(\gamma))_{n=1}^{\infty}$ converges $\sigma(JT^{***}, JT^{**})$ to $\chi_{k(\gamma)} - \frac{1}{2}\chi_{\lambda(\gamma)}$.

Hence, for $\gamma \notin A$, $\operatorname{dist}(S_A(\gamma), Z_A) \leqq \| \chi_{k(\gamma)} - (\frac{1}{2})\chi_{\lambda(\gamma)} \| \leqq \sqrt{2}/2$.

So the function $\varphi : K \to \mathbf{R}_+$ equals 1 on $\psi^{-1}(A)$ and is less than or equal to $\sqrt{2}/2$ on $\psi^{-1}(\Delta \setminus A)$. As A is not μ-measurable, φ is not $\hat{\mu}$-measurable. ■

REMARK 7.3. The above Banach space Z_A also gives a counterexample to a question raised by K. Musiał at the 15th winter school of the Cech Academy of Science in Šrni (January 1987) on extendability of Pettis-integrable functions:

Let $A \subseteq \Delta$ be as above (i.e., $\mu^*(A) = 1$, $\mu_*(A) = 0$) and let $(A, \tilde{\Sigma}, \tilde{\mu})$ be the measure space induced by the outer measure μ^* on the trace $\tilde{\Sigma}$ of the μ-measurable subsets Σ of Δ on A. If $k : A \to \Delta$ is the canonical embedding, then clearly $k(\tilde{\mu}) = \mu$ and the operator $l : L^1(\Delta, \mu) \to L^1(A, \tilde{\mu})$ defined by $l(f) = f \circ k$ is an isometric isomorphism between $L^1(\mu)$ and $L^1(\tilde{\mu})$.

PROPOSITION 7.4. *With the above notation there is a Banach space X and an operator $S : L^1(A, \tilde{\mu}) \to X$ which is representable by a Pettis-integrable function $\Phi : A \to X$ but such that the operator $S \circ l : L^1(\Delta, \mu) \to X$ is not representably a Pettis-integrable function $\Psi : \Delta \to X$.*

PROOF. It suffices to let $X = Z_A$ and $S = T_A \circ l^{-1}$, where T_A and Z_A are defined above.

Let Φ be the restriction to A of the function F defined at the beginning of Section 7. Clearly Φ takes its values in Z_A, is Pettis integrable, and represents S.

On the other hand, consider the operator $i_A \circ S \circ l : L^1(\Delta, \mu) \rightarrow JT^*$. This operator is represented by a Pettis-integrable function, namely $F : \Delta \rightarrow JT^*$.

As JT is separable, we conclude that any Pettis-integrable function $G : \Delta \rightarrow JT^*$ representing $i_A \circ S \circ l$ equals F μ-almost everywhere. In particular, any such G must take its values in $JT^* \setminus i_A(Z_A)$ on a set of μ-outer measure 1.

Hence there cannot be a Pettis-integrable function Ψ representing $S \circ l : L^1(\Delta, \mu) \rightarrow Z_A$ as $i_A \circ \Psi$ then would represent $i_A \circ S \circ l$ and take its values in $i_A(Z_A)$.

This contradiction finishes the proof. ∎

References

[B1] J. Bourgain, *A geometric characterisation of the RNP in Banach spaces*, Compositio Math. **36** (1978), 3–6.

[B2] J. Bourgain, *Dentability and finite-dimensional decompositions*, Studia Math. **67** (1980), 135–148.

[B3] J. Bourgain, *Sets with the Radon–Nikodým property in conjugate Banach spaces*, Studia Math. **66** (1980), 291–297.

[Br] R. Brackebusch, *James space on general trees*, J. Funct. Anal., to appear.

[D–U] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys, **15**, Am. Math. Soc., Providence, 1977.

[D–Z] R. Deville and V. Zizler, *Farthest points in ω^*-compact sets*, preprint.

[G–G–M–S] N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Am. Math. Soc., No. 378 (1987).

[G–M] N. Ghoussoub and B. Maurey, *H_δ embedding and optimisation on G_δ sets*, Mem. Am. Math. Soc., No. 349 (1986).

[Go] G. Godefroy, *Metric characterization of first Baire class linear forms and octahedral norms*, preprint.

[Go–M] G. Godefroy and B. Maurey, *Normes lisses et anguleuses sur les espaces de Banach séparables*, unpublished.

[H] R. Haydon, *Some more characterisations of Banach spaces not containing l^1*, Proc. Camb. Phil. Soc. **80** (1976), 269–276.

[J] R. James, *A separable somewhat reflexive Banach space with non-separable dual*, Bull. Am. Math. Soc. **80** (1974), 738–743.

[L–S] J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain l^1 and whose duals are not separable*, Studia Math. **54** (1974), 81–105.

[L–L–T] Bor-Lin Lin, Pei-Kee Lin and S. Troyanski, *A characterisation of denting points of a closed bounded, convex set*, Proc. Am. Math. **345** (1988), 526–528.

[O–R] E. Odell and H. P. Rosenthal, *A double dual characterization of separable Banach spaces containing l^1*, Isr. J. Math. **20** (1975), 375–384.

[P] A. Pełczyński, *On Banach spaces containing $L^1(\mu)$*, Studia Math. **30** (1968), 231–246.

[R] H. P. Rosenthal, *On non-norm-attaining functionals and the equivalence of the weak-star-KMP with the RNP*, Longhorn Notes 1985–86, The University of Texas, Austin.

[S1] W. Schachermayer, *Some more remarkable properties of the James tree space*, to appear.

[S2] W. Schachermayer, *The Radon–Nikodým and the Krein–Milman property are equivalent for stongly regular sets*, Trans. Am. Math. Soc. **303** (1987), 673–687.

[St1] C. Stegall, *The Radon Nikoým property in conjugate spaces*, Trans. Am. Math. Soc. **206** (1975), 213–223.

[St2] C. Stegall, *The RNP in Banach spaces I*, Vorlesungen aus dem Fachbereich Mathematik der University Essen, Heft 10, Essen, 1983.

[T] M. Talagrand, *Pettis integral and measure theory*, Mem. Am. Math. Soc., No. 307 (1984).

[W] E. Werner, *Non dentable solid subsets in Banach lattices failing RNP, and applications to renormings*, Proc. Am. Math. Soc., to appear.